Second-Order Abstract Interpretation via Kleene Algebra

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Abstract Interpretation

Cousot & Cousot 79

- Static derivation of information about the execution state at various points in a program
- Comes in various flavors
  - type inference
  - dataflow analysis
  - set constraints
- Applications
  - code optimization
  - verification
  - generating proof artifacts for PCC

Standard Approach

- Start with the control flow graph of the program to be analyzed
- Propagate known information forward – possible values of variables or types
- Compute a join at confluence points
- Standard method is called the worklist algorithm
- The process is a bit like running the program on abstract values, hence the name abstract interpretation

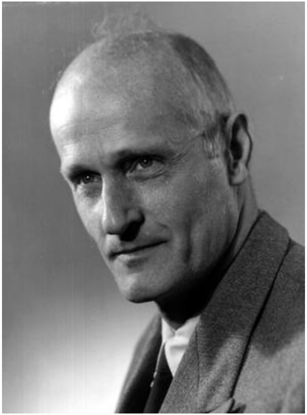
Types or Abstract Values

- Represent sets of values
  - statically derivable
  - conservative approximation
- Form a partial semilattice
  - higher = less specific
  - join does not exist = type error
- Often, abstract values are associated with invariants

This Talk

- A general mechanism for abstract interpretation and dataflow analysis based on Kleene algebra
- May improve performance over standard worklist algorithm when the semilattice of types is small
- Illustration of the method in the context of Java bytecode verification

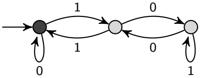
Kleene Algebra (KA)



Stephen Cole Kleene
(1909–1994)

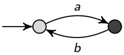
$(0 + 1(01^*0)^*1)^*$

{multiples of 3 in binary}



$(ab)^*a = a(ba)^*$

$\{a, aba, ababa, \ldots\}$



$(a + b)^* = a^*(ba^*)^*$

{all strings over $\{a, b\}$}

Foundations of the Algebraic Theory

John Horton Conway
(1937–)

J. H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, London, 1971.

Axioms of KA

Idempotent Semiring Axioms

$$p + (q + r) = (p + q) + r \qquad p(qr) = (pq)r$$
$$p + q = q + p \qquad 1p = p1 = p$$
$$p + 0 = p \qquad p0 = 0p = 0$$
$$p + p = p$$
$$p(q + r) = pq + pr \qquad a \leq b \overset{\text{def}}{\iff} a + b = b$$
$$(p + q)r = pr + qr$$

Axioms for *

$$1 + pp^* \leq p^* \qquad q + px \leq x \;\Rightarrow\; p^*q \leq x$$
$$1 + p^*p \leq p^* \qquad q + xp \leq x \;\Rightarrow\; qp^* \leq x$$

Significance of the * Axioms

$$1 + pp^* \leq p^* \;\Rightarrow\; q + pp^*q \leq p^*q$$
$$q + px \leq x \;\Rightarrow\; p^*q \leq x$$

p^*q is the least x such that $q + px \leq x$

Standard Model

Regular sets of strings over Σ

$$
\begin{aligned}
A + B &= A \cup B \\
AB &= \{xy \mid x \in A,\ y \in B\} \\
A^* &= \bigcup_{n \geq 0} A^n \quad = \quad A^0 \cup A^1 \cup A^2 \cup \cdots \\
1 &= \{\varepsilon\} \\
0 &= \varnothing
\end{aligned}
$$

This is the free KA on generators Σ

Relational Models

Binary relations on a set X

For $R, S \subseteq X \times X$,

$$
\begin{aligned}
R + S &= R \cup S \\
RS &= R \circ S \;=\; \{(u,v) \mid \exists w\ (u,w) \in R,\ (w,v) \in S\} \\
R^* &= \text{reflexive transitive closure of } R \\
&= \bigcup_{n \geq 0} R^n \;=\; R^0 \cup R^1 \cup R^2 \cup \cdots \\
1 &= \text{identity relation} \;=\; \{(u,u) \mid u \in X\} \\
0 &= \varnothing
\end{aligned}
$$

KA is complete for the equational theory of relational models

Other Models

- Trace models used in semantics
- $(\min, +)$ algebra used in shortest path algorithms
- $(\max, \cdot)$ algebra used in coding
- Convex sets used in computational geometry [Iwano & Steiglitz 90]

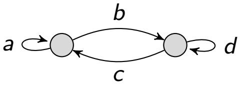
Matrices over a KA form a KA

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{bmatrix}$$



Systems of Affine Linear Inequalities

Theorem

Any system of n linear inequalities in n unknowns has a unique least solution

$$\begin{aligned}
q_1 + p_{11}x_1 + p_{12}x_2 + \cdots p_{1n}x_n &\leq x_1 \\
&\vdots \\
q_n + p_{n1}x_1 + p_{n2}x_2 + \cdots p_{nn}x_n &\leq x_n
\end{aligned}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} & & \\ & P = p_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Least solution is $P^* q$

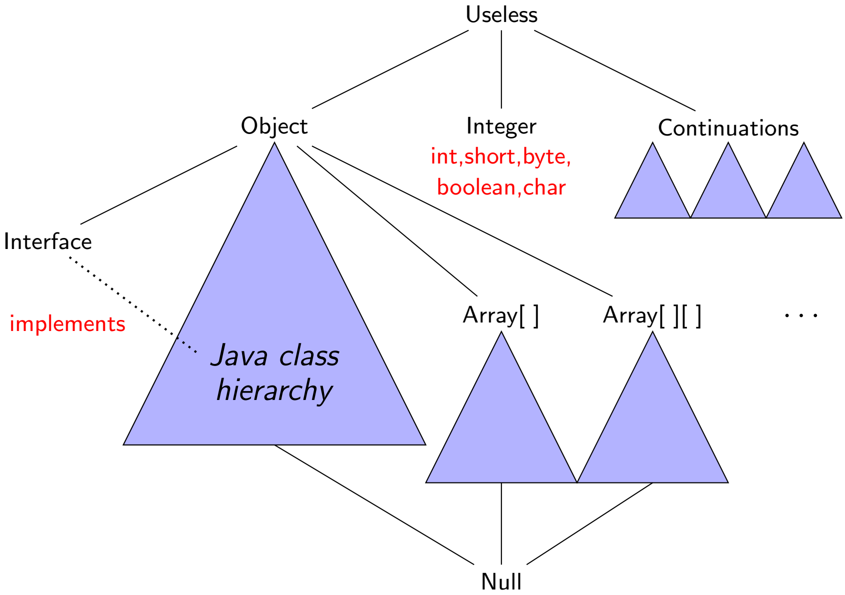
Proof Artifacts

An independently verifiable representation of the proof

$x \le y \Rightarrow x* \le y*$

```
λx,y.λP0.(trans< [y=x*;1 x=x* z=y*] (=< [x=x* y=x*;1]
(sym [x=x*;1 y=x*] (id.R [x=x*])),*R [x=x y=1 z=y*]
(trans< [y=1 + y;y* x=x;y* + 1 z=y*]
(trans< [y=y;y* + 1 x=x;y* + 1 z=1 + y;y*]
(mono+R [x=x;y* y=y;y* z=1] (mono.R [x=x y=y z=y*] P0),
=< [x=y;y* + 1 y=1 + y;y*] (commut+ [x=y;y* y=1])),
=< [x=1 + y;y* y=y*] (unwindL [x=y])))))
```

Example: Java Bytecode Verification

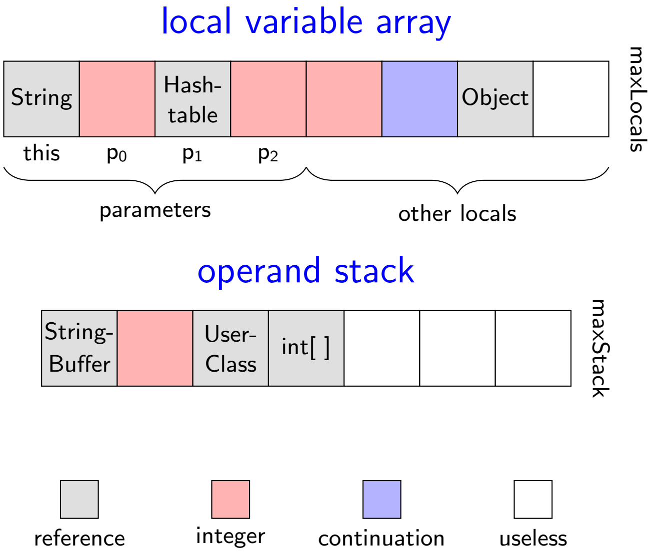


Example: Java Bytecode Verification

Typical bytecode instructions:

`iload 3`	load an int from local 3, push on the operand stack
`istore 3`	pop an int from the operand stack, store in local 3
`iadd`	add the two ints on top of the stack, leave result on stack
`aload 4`	load a ref from local 4, push on the operand stack
`astore 4`	pop a ref from the operand stack, store in local 4
`swap`	swap the two values on top of the stack (polymorphic)

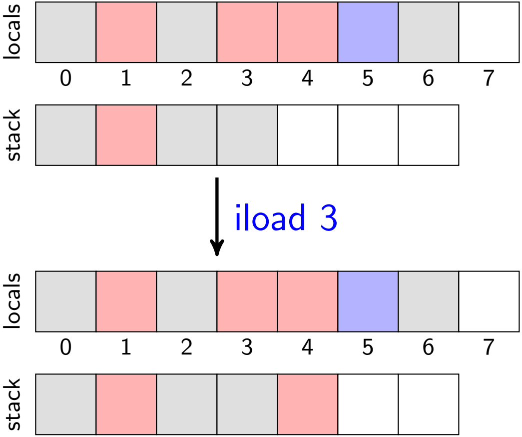
Example: Java Bytecode Verification



A Directed Graph

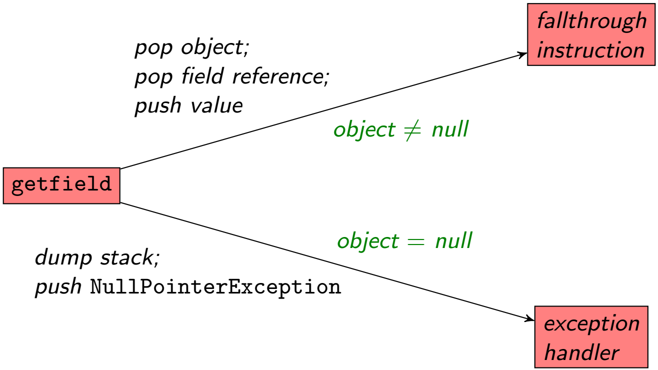
- Vertices are instruction instances
- Edges to successor instructions, statically determined
  - fallthrough
  - jump targets
  - exception handlers
- Edges labeled with transfer functions
  - partial functions types $\rightarrow$ types
  - models abstract effect of instruction
  - domain of definition gives precondition for safe execution
  - different successors may have different transfer functions

Example of a Transfer Function

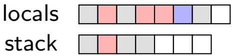


- Preconditions for safe execution
  - local 3 is an integer
  - stack is not full
- Effect
  - push integer in local 3 on stack

Different exiting edges ⇒ different transfer functions

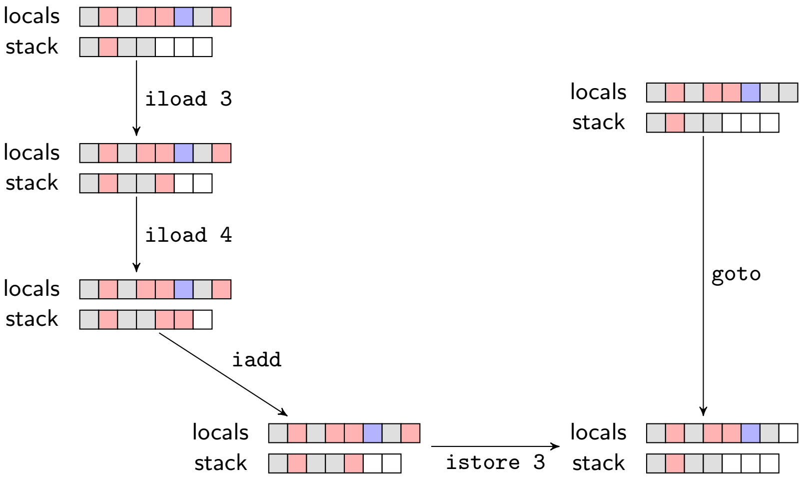


Abstract Interpretation

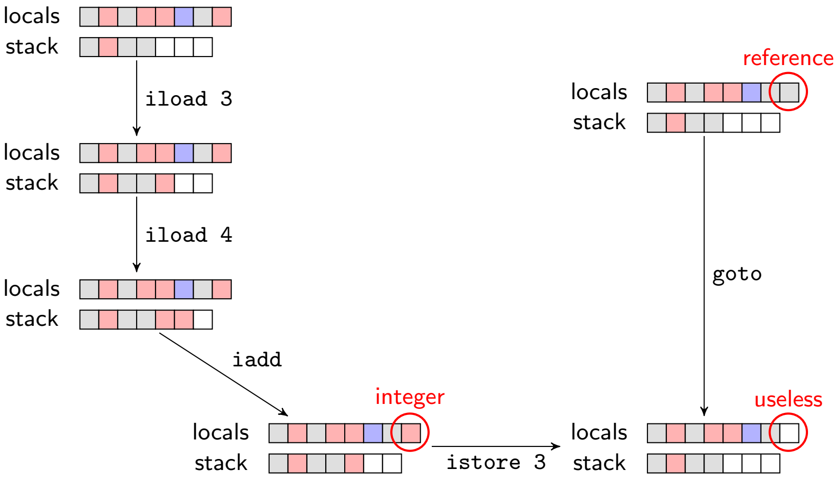


- Annotate each vertex with a type
  - reflects best knowledge of the state immediately prior to execution of the instruction
  - must satisfy preconditions of exiting transfer functions
- Annotation of the entry instruction is determined by the declared type of the method
- Annotation of other instructions = join of values of transfer functions applied to predecessors annotations
- Want least fixpoint = best conservative approximation

Example

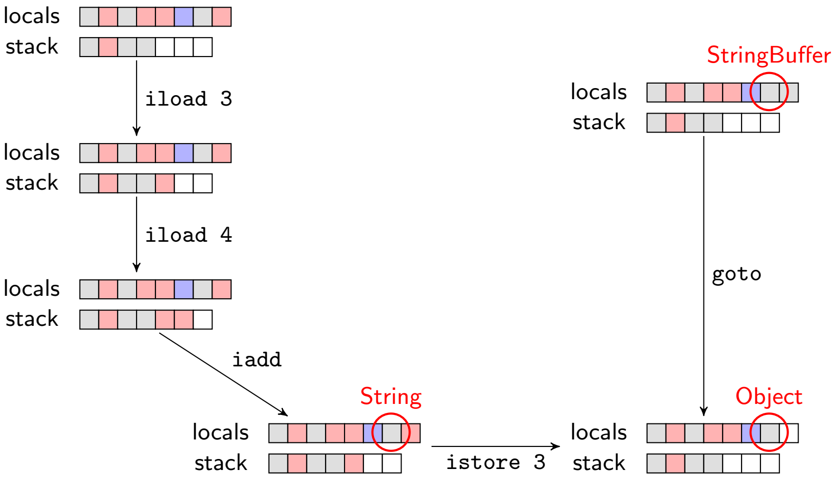

locals
stack
iload 3
locals
stack
iload 4
locals
stack
iadd
locals
stack
istore 3
locals
stack
goto
locals
stack

Example



locals
stack
iload 3
locals
stack
iload 4
locals
stack
iadd
integer
locals
stack
istore 3
reference
locals
stack
goto
useless
locals
stack

Example



locals
stack
iload 3
locals
stack
iload 4
locals
stack
iadd
String
locals
stack
istore 3
StringBuffer
locals
stack
goto
Object
locals
stack

Basic Worklist Algorithm

- Annotate entry instruction according to declared type of the method, put on worklist
  - first $n + 1$ locals contain `this`, method parameters
  - stack is empty
- Repeat until worklist is empty:
  - remove next instruction from worklist
  - for each exiting edge:
    - apply transfer function on that edge to current annotation
    - update successor annotation – join of transfer function value and current successor annotation
    - join does not exist $\Rightarrow$ type error
    - if successor changed, put on worklist

An Application of Kleene Algebra

- Idea: avoid retracing of long cycles by symbolic composition of transfer functions
- Elements of the Kleene algebra are (typed) transfer functions
  - multiplication = typed composition
  - addition = join in the type semilattice
- Least fixpoint calculation involves computing the * of an $m \times m$ matrix, where m is the size of a cutset (set of vertices breaking all cycles)

Semilattices and the ACC

- Let $(L, +, \bot)$ be a semilattice satisfying the ascending chain condition (ACC)

$$x + (y + z) = (x + y) + z \qquad x + \bot = x$$
$$x + y = y + x \qquad x + x = x$$

- ACC = no infinite ascending chains in L
- Implies that L contains a maximum element $\top$
- Elements of L represent dataflow information
  - lower = more information
  - higher = less information
  - $\top$ = no information

A Partial Order

- There is a natural partial order

$$x \leq y \overset{\text{def}}{\iff} x + y = y$$

- $x + y$ is the least upper bound of x and y with respect to $\leq$

Transfer Functions

- Transfer functions are modeled as strict, monotone functions $f : L \to L$
  - monotone: $x \leq y \Rightarrow f(x) \leq f(y)$
  - strict: $f(\bot) = \bot$
- Examples: $0 = \lambda x.\bot$, $1 = \lambda x.x$
- The domain of f is

$$\mathrm{dom}\, f = \{x \in L \mid f(x) \neq \top\}$$

- monotonicity implies $\mathrm{dom}(f)$ closed downward under $\leq$

Join

- Define a join operation on transfer functions:

$$(f+g)(x) = f(x) + g(x)$$

- $0 = \lambda x.\bot$ is a two-sided identity for $+$

$$((\lambda x.\bot) + g)(x) = \bot + g(x) = g(x)$$

- idempotent $f + f = f$, thus we have a natural partial order

$$f \leq g \overset{\text{def}}{\iff} f + g = g$$

- upper semilattice with least element $0 = \lambda x.\bot$

Composition

Write $f;g$ for the ordinary functional composition $g \circ f = \lambda x.g(f(x))$

- $x \in \mathrm{dom}(f;g)$ iff $x \in \mathrm{dom}\, f$ and $f(x) \in \mathrm{dom}\, g$, and

$$(f;g)(x) = g(f(x))$$

- $\lambda x.x$ is a two-sided identity for composition

$$f;(\lambda x.x) = (\lambda x.x);f = f$$

- composition is monotone

$$f \leq g \Rightarrow f;h \leq g;h \qquad\qquad f \leq g \Rightarrow h;f \leq h;g$$

- $0 = \lambda x.\bot$ is a two-sided annihilator

$$(\lambda x.\bot);f = f;(\lambda x.\bot) = \lambda x.\bot$$

Distbutive Laws

Composition distributes over + on the left

$$f ; (g + h) = f ; g + f ; h$$

but not on the right; however

$$f ; h + g ; h \leq (f + g) ; h$$

due to monotonicity

Star

$f^* : L \to L$ is the function

$$f^*(x) = \text{the least } y \text{ such that } x + f(y) \leq y$$

This exists, since f is monotone and the ACC holds, so the monotone sequence

$$x,\ x + f(x),\ x + f(x + f(x)),\ \ldots$$

converges after a finite number of steps

The convergence is not necessarily uniformly bounded in x

Counterexample: take $L = \mathbb{N} \cup \{\infty\}$, join $=$ min, $f(x) = \infty$ if $x = \infty$, $x - 1$ if $x \geq 1$, and 0 if $x = 0$

Modeling Transfer Functions

We define a left-handed Kleene algebra to be a structure that satisfies all the axioms of Kleene algebra, except

- we only require the left-handed * axioms and
- only right subdistributivity

Let K be the set of monotone strict functions $L \to L$.

Theorem

The structure $(K, +, \cdot, {}^*, 0, 1)$ *is a left-handed Kleene algebra.*

Theorem

The set of $n \times n$ *matrices over a left-handed Kleene algebra with the usual matrix operations is again a left-handed Kleene algebra.*

Dataflow as Matrix *

- Let $S = \{\text{vertices of the dataflow graph}\}$
- Let $E =$ the $S \times S$ matrix whose $(s, t)^{\text{th}}$ entry is the transfer function labeling edge (s, t)
- Let s_0 be the entry point of the method, $\theta_0 \in L$ its initial label
- $E^*(s, t)$ is the join of all labels on paths from s to t

Theorem

$E^*(s_0, t)(\theta_0)$ is the least fixpoint dataflow annotation of t. It is the same labeling as that produced by the worklist algorithm.

An Example

```
if (b) x = y + 1;
else x = z;
```

```
      (if b then α)
      iload 5     //load z     }
      istore 3    //save x     } else
      goto β                   }
α:    iload 4     //load y     }
      iconst 1    //load 1     }
      iadd                     } then
      istore 3    //save x     }
β:    ...
```

An Example

```
if (b) x = y + 1;
else x = z;
```

	(if b then α)			
	iload 5	//load z	*else*	(iload 5;
	istore 3	//save x		istore 3)
	goto β			+
α:	iload 4	//load y	*then*	(iload 4;
	iconst 1	//load 1		iconst 1;
	iadd			iadd;
	istore 3	//save x		istore 3)
β:	...			

An Example

`x = z;`	*precondition*	*effect*
`iload 5`	5:int depth $<$ maxStack-1	stack $=$ int::$\cdots$, $\partial = 1$
`istore 3`	int::stack	$\partial = -1$ 3:int

An Example

`x = z;`	*precondition*	*effect*
`iload 5`	5:int depth $<$ maxStack-1	stack $=$ int::$\cdots$, $\partial = 1$
`istore 3`	int::stack	$\partial = -1$ 3:int

compose

`iload 5` `istore 3`	5:int depth $<$ maxStack-1	$\partial = 0$ 3:int

An Example

`x = y+1;`	*precondition*	*effect*
`iload 4`	4:int depth $<$ maxStack-1	stack $=$ int::$\cdots$, $\partial = 1$
`iconst 1`	depth $<$ maxStack-1	stack $=$ int::$\cdots$, $\partial = 1$
`iadd`	int::int::stack	$\partial = -1$
`istore 3`	int::stack	$\partial = -1$ 3:int

An Example

`x = y+1;`	*precondition*	*effect*
`iload 4`	4:int depth $<$ maxStack-1	stack $=$ int::$\cdots$, $\partial = 1$
`iconst 1`	depth $<$ maxStack-1	stack $=$ int::$\cdots$, $\partial = 1$
`iadd`	int::int::stack	$\partial = -1$
`istore 3`	int::stack	$\partial = -1$ 3:int

compose

`iload 4`
`iconst 1`
`iadd`
`istore 3`
	4:int
	depth $<$ maxStack-2
		$\partial = 0$
		3:int

An Example

	precondition	*effect*
`iload 5`	5:int	$\partial = 0$
`istore 3`	depth $<$ maxStack−1	3:int
`iload 4`	4:int	$\partial = 0$
`iconst 1`	depth $<$ maxStack−2	3:int
`iadd`		
`istore 3`		

An Example

	precondition	*effect*
`iload 5`	5:int	$\partial = 0$
`istore 3`	depth $<$ maxStack–1	3:int
`iload 4`	4:int	$\partial = 0$
`iconst 1`	depth $<$ maxStack–2	3:int
`iadd`		
`istore 3`		

join

`iload 5`		
`istore 3`		
+	4:int, 5:int	$\partial = 0$
`iload 4`	depth $<$ maxStack–2	3:int
`iconst 1`		
`iadd`		
`istore 3`		

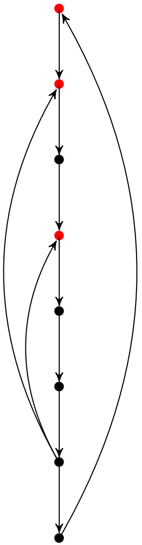
Dataflow as Matrix *

Theorem
$E^*(s_0, t)(\theta_0)$ is the least fixpoint dataflow annotation of t. It is the same labeling as that produced by the worklist algorithm.

- Problem: E is huge (but sparse)
- Solution: find a small cutset

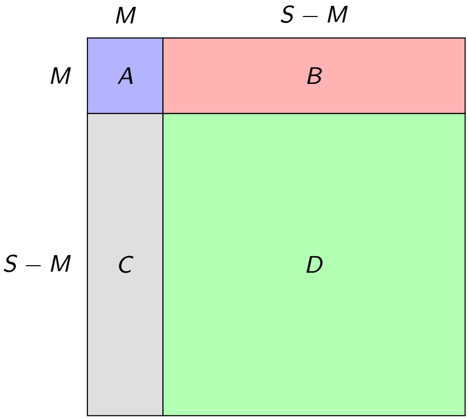
Cutsets

- A cutset (a.k.a. feedback vertex set) is a set M of vertices breaking all directed cycles
- To compute the least fixpoint labeling efficiently, need to identify a small cutset
- Finding a minimal cutset is NP-complete, but polynomial time for reducible graphs
- In practice, take $M = \{\text{targets of back edges}\}$



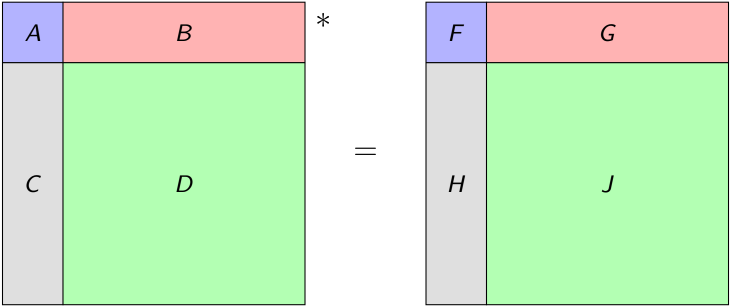
Dataflow as Matrix *

- Partition E into submatrices indexed by M and $S - M$, where M is the cutset



- That M is a cutset is reflected algebraically by the property $D^n = 0$, where $n = |S - M|$

Dataflow as Matrix *



where

$$F = (A + BD^* C)^* \qquad G = FBD^*$$

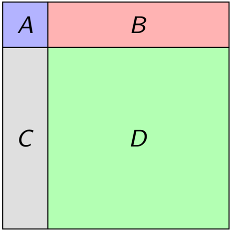
$$H = D^* CF \qquad J = D^* + D^* CFBD^*$$

Dataflow as Matrix *

- $D^n = 0 \Rightarrow D^* = (I + D)^{n-1}$
- The $M \times M$ submatrix of E^* is

  $$(A + BD^*C)^* = (A + B(I + D)^{n-1}C)^*$$

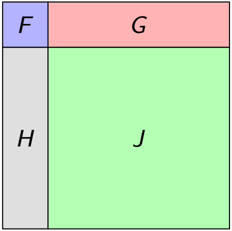
- If s, t are cutpoints, the $(s, t)^{\mathrm{th}}$ entry of $B(I + D)^{n-1}C$ is the join of all paths $s \to t$ containing no other cutpoint
- Compute by repeated squaring or a variant of Dijkstra



Dataflow as Matrix *

- $F = (A + B(I + D)^{n-1}C)^*$ is much smaller than E
- The other submatrices of E^* can be described in terms of this matrix

$$G = FBD^*$$
$$H = D^*CF$$
$$J = D^* + HG$$



Finding Small Cutsets

Efficiency depends on finding a small cutset = set of nodes intersecting every directed cycle

- finding a minimum cutset is *NP*-complete
- Ptime for reducible graphs [Garey & Johnson 79]
- bytecode programs compiled from Java source are typically reducible
- in practice, take targets of back edges

How big are cutsets in practice?

Finding Small Cutsets

Efficiency depends on finding a small cutset = set of nodes intersecting every directed cycle

- finding a minimum cutset is *NP*-complete
- Ptime for reducible graphs [Garey & Johnson 79]
- bytecode programs compiled from Java source are typically reducible
- in practice, take targets of back edges

How big are cutsets in practice?

- analyzed 537 Java programs
- median cutset size = 2.1% of total program size
- all except 5 programs $< 5\%$
- largest program analyzed was 2668 instructions with 5 cutpoints = 0.2%

A Pipe Dream

- Many instructions have preconditions for safe execution (e.g., array, pointer dereference). Compilers should either:
  - insert a runtime type check, or
  - optimize away the check, but provide a proof of correctness of the optimization

- Programmer should be able to specify such preconditions, and they should behave the same way as the built-in ones

```
if (h.containsKey(key)) {
   data = h.get(key);
} else {
   data = new Data();
   h.put(key,data);
}
```

```
data = h.get(key);
if (data == null) {
   data = new Data();
   h.put(key,data);
}
```

```
data = h.get(key);
```

```
if (h.containsKey(key)) {
   data = h.get(key);
} else {
   data = new Data();
   h.put(key,data);
}

data = h.get(key);
if (data == null) {
   data = new Data();
   h.put(key,data);
}

assert h.containsKey(key);
data = h.get(key);
```

Built-in Preconditions

```
x = obj.data;
```

```
x = a[i];
```

Compiler will either

- omit runtime check but supply a proof, or
- insert runtime check and throw exception on failure (`NullPointerException` or `ArrayIndexOutOfBoundsException`, resp.)

Built-in Preconditions

```
assert obj != null;
x = obj.data;

assert 0 <= i && i < a.length;
x = a[i];
```

Compiler will either

- omit runtime check but supply a proof, or
- insert runtime check and throw exception on failure (`NullPointerException` or `ArrayIndexOutOfBoundsException`, resp.)

Programmer-Defined

```
assert h.containsKey(key);
data = h.get(key);
```

Compiler will either

- omit runtime check but supply a proof, or
- insert runtime check and throw `InvalidAssertionException` on failure

Conclusion

Summary

- A general mechanism for second-order abstract interpretation based on Kleene algebra
  - may improve performance over standard worklist algorithm when the semilattice of types is small - $O(m^3 + nm)$ vs $O(nd)$
- Proved soundness and completeness of the method
- Illustrated the method in the context of Java bytecode verification

Possible next steps

- Implement and compare experimentally to the standard worklist algorithm as specified in the Java VM specification
- Second-order method is amenable to parallelization, whereas the standard worklist method is inherently sequential
  - application of a transfer function requires knowledge of its inputs
  - compositions can be computed without knowing their inputs

Thanks!