Linear Algebra, Boolean Rings and Resolution?

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- functional-level
  - high-level descriptions of algorithms assume infinite memory
  - for instance infinite tape, integers, functional languages, ...
- word-level
  - 32 or 64 bit systems
  - modular arithmetic instead of integer arithmetic
  - pointer arithmetic
- bit-level
  - HW designs synthesized to bit-level
  - processors all work on the bit-level

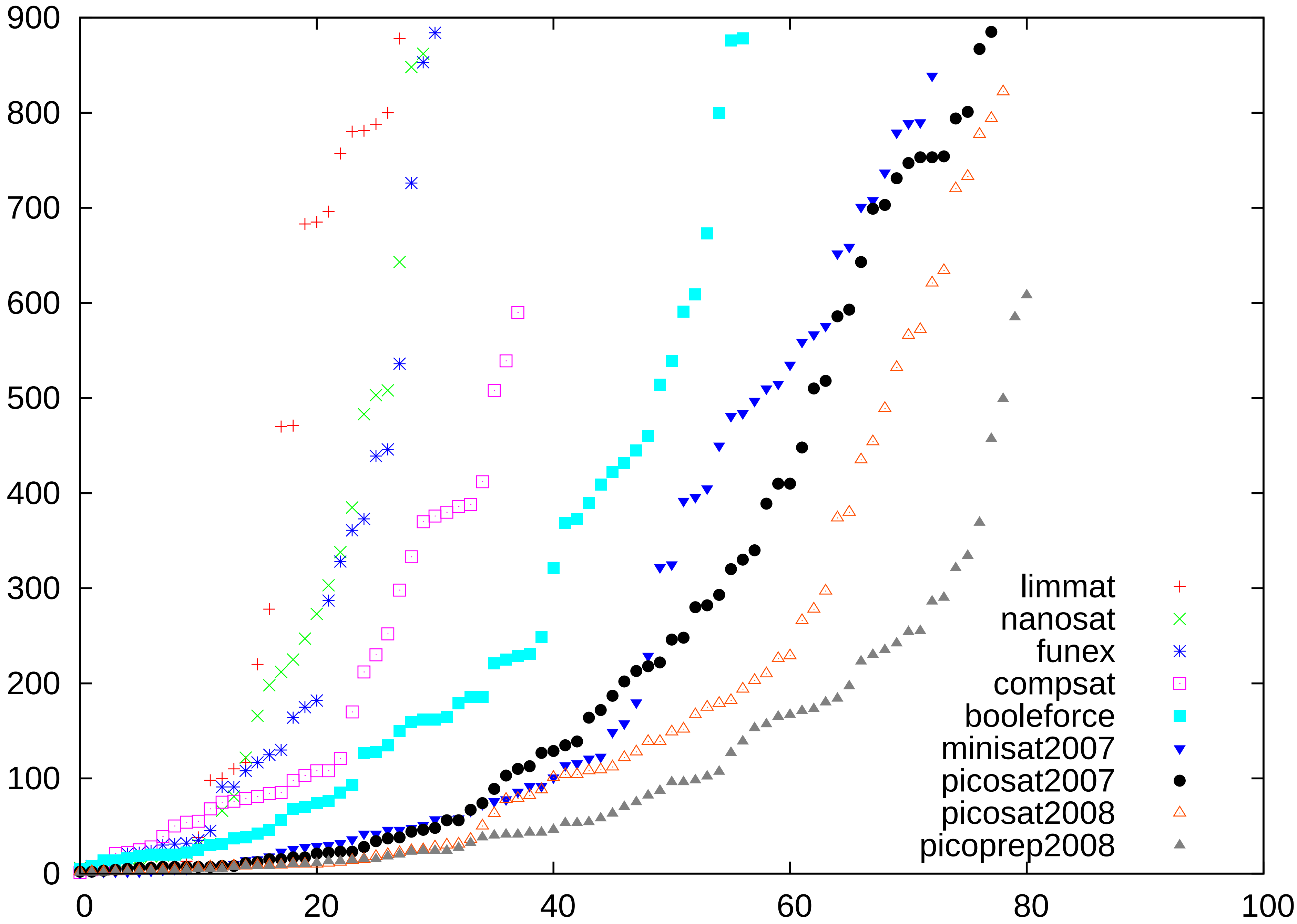
- equivalence checking (HW)
  - heavy (mostly) automatic optimizations on the bit-level
  - comparison with “golden” original implementation
- cryptanalysis (particularly of stream-cyphers)
  - described on the word-level
  - many bit-level operations, e.g. LFSR, AND-gates, XOR-gates
- verifying modular arithmetic in SW

  “Nearly All Binary Searches and Mergesorts are Broken”

  ```
  int l, r, m; ...  m = (l + r)/2; ...
  ```

- DPLL (still!) plus
  - learning: GRASP, RelSAT, SATO
  - VSIDS decision heuristics: Chaff, MiniSAT, PicoSAT, ...
  - ... and many more (important) optimizations:
    restarts, pre-processing, data structures
- driven by yearly SAT competition / SAT race and **many** applications
- extensions to *satisfiability modulo theories* (SMT)
- **the** formal core technology in industry:
  equivalence checking, bounded and unbounded model checking, synthesis
  test case generation, coverage, consistency checking, configuration ...

Cactus Plot for SAT'06 Race Instances



- equivalence checking of arithmetic circuits (on the bit-level) is very difficult
  - for instance associativity: `x * (y * z)` vs `(x * y) * z`
  - needs four 32x32 to 32 bit multipliers after "bit-blasting"
  - again: we need to reason on the bit-level!!

- breaking a stream cypher also needs bit-level reasoning
  - long XOR-chains are bad for standard SAT solvers
  - example: compute parity with two structural different circuits

- Why not use algebraic methods for boolean rings?

- $+$ = XOR $\cdot$ = AND $K = \mathbb{Z}_2 = \{0,1\}$

- SAT usually works on conjunctive normal form (CNF)

  – we can either transform CNF into Ideal

  $(\neg a \vee b) \wedge (\neg a \vee c) \wedge (a \vee \neg b \vee \neg c)$ satisfiable

  iff

  $1 + a(b+1) = 1, \quad 1 + a(c+1) = 1, \quad 1 + (a+1)bc = 1$ solvable

  iff

  $\langle ab + a, ac + a, abc + b + c \rangle \neq \langle 1 \rangle$

  with

  $\neg a = 1 + a, \quad a \vee b = \neg(\neg a \wedge \neg b) = 1 + (a+1)(b+1) = ab + a + b$

  – or apply similar transformation/encoding of original problem (Tseitin)

- linear algebra
  - Gaussian elimination
  - provides a generalization of various techniques for "equivalence reasoning"
  - can still not be applied blindly (SAT solvers handle million of variables)
  - similar integration as in SMT solvers? DPLL (LA($\mathbb{Z}_2$))
- polynomials
  - computing Gröbner bases with Buchberger's algorithm
  - brute force too expensive (similar problems as DP algorithm)
  - refutational completeness useless in practice
  - useful for preprocessing (?!)

- given two square $n \times n$ matrices A, B over $\mathbb{Z}_2$, then $AB = 1 \Rightarrow BA = 1$

- algebraic bit-level encoding: n^2 polynomials for LHS, n^2 polynomials for RHS
  - compute Gröbner basis for LHS
  - check that each of the RHS polynomials is contained in the generated ideal

- CNF encoding: circuits of size $O(n^3)$ for both LHS and RHS

- benchmark in the crafted category of the SAT solver competition (linvrinv)
  - SAT solvers: $n = 4$: seconds $n = 5$: 800 - 2000 seconds $n = 6$: unsolved
  - Singular: $n = 4$: seconds $n = 5, 6$: unsolved

```
ring r = 2, (
  x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15,x16,x17,x18,x19,x20,
  x21,x22,x23,x24,x25,x26,x27,x28,x29,x30,x31,x32,x33,x34,x35,x36,x37,x38,
  x39,x40,x41,x42,x43,x44,x45,x46,x47,x48,x49,x50), dp;

ideal I = [
  x1*x2+x3*x12+x5*x22+x7*x32+x9*x42+1,x1*x4+x3*x14+x5*x24+x7*x34+x9*x44,
  x1*x6+x3*x16+x5*x26+x7*x36+x9*x46,x1*x8+x3*x18+x5*x28+x7*x38+x9*x48,
  x1*x10+x3*x20+x5*x30+x7*x40+x9*x50,x11*x2+x13*x12+x15*x22+x17*x32+x19*x42,
  x11*x4+x13*x14+x15*x24+x17*x34+x19*x44+1,
  x11*x6+x13*x16+x15*x26+x17*x36+x19*x46,
  x11*x8+x13*x18+x15*x28+x17*x38+x19*x48,
  x11*x10+x13*x20+x15*x30+x17*x40+x19*x50,
  x21*x2+x23*x12+x25*x22+x27*x32+x29*x42,x21*x4+x23*x14+x25*x24+x27*x34+x29*x44,
  x21*x6+x23*x16+x25*x26+x27*x36+x29*x46+1,
  x21*x8+x23*x18+x25*x28+x27*x38+x29*x48,
  x21*x10+x23*x20+x25*x30+x27*x40+x29*x50,x31*x2+x33*x12+x35*x22+x37*x32+x39*x42,
  x31*x4+x33*x14+x35*x24+x37*x34+x39*x44,x31*x6+x33*x16+x35*x26+x37*x36+x39*x46,
  x31*x8+x33*x18+x35*x28+x37*x38+x39*x48+1,
  x31*x10+x33*x20+x35*x30+x37*x40+x39*x50,
  x41*x2+x43*x12+x45*x22+x47*x32+x49*x42,x41*x4+x43*x14+x45*x24+x47*x34+x49*x44,
  x41*x6+x43*x16+x45*x26+x47*x36+x49*x46,x41*x8+x43*x18+x45*x28+x47*x38+x49*x48,
  x41*x10+x43*x20+x45*x30+x47*x40+x49*x50+1];

ideal J = groebner (I);
```

- Gaussian Elimination in $\mathbb{Z}_2$ can be simulated by (RO)BDD operations
  - BDD to store a linear equation is linear in the number n of variables
  - XOR operation on BDDs for lin. equations has linear complexity in n
  - in general, BDD operations are in $O(n^2)$
- BDD operations can be simulated by extended resolution [SinzBiere-CSR'06]
  - extension rule: add literal equation $a = b \wedge c$ with fresh a
  - extended resolution is the most powerful bit-level proof system
  - proof linear in the number of recursive BDD computation steps
  - proofs are used in many applications

- same idea does not lift to polynomials:
  - ROBDD size quadratic in the size of the represented polynomial (?)
  - complexity of operations totally unclear
- conjecture:
  - ROBDDs can **not** simulate Buchberger's algorithm linearly
  - unclear whether other BDD variants allow linear simulations
- challenge
  - directly generate (extended) resolution proofs from polynomial reasoning

Summary

- a case for bit-level reasoning ...

- SAT solvers made and are still making tremendous progress

- difficult: arithmetic on the bit-level and cryptanalysis

- Stephen Cook's SAT'04 challenge captures the essence of this problem

- algebraic methods (out of the box) provide no silver bullet

- we need combinations of algebraic methods with SAT on the bit-level

- extensions to word-level (bit-vector) decisions procedures ? $\Rightarrow$ Boolector