QBF in Formal Verification

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Alpine Verification Meeting 2005

Laussanne, Switzerland
6. October 2006

Quantified Boolean Formulae (QBF)

A. Biere, FMV, JKU Linz

- propositional logic (SAT $\subseteq$ QBF)
  - constants $0, 1$
  - operators $\wedge, \neg, \rightarrow, \leftrightarrow, \ldots$
  - variables $x, y, \ldots$ over boolean domain $\mathbb{B} = \{0, 1\}$
- quantifiers over boolean variables
  - valid $\forall x[\exists y[x \leftrightarrow y]]$ (read $\leftrightarrow$ as $=$)
  - invalid $\exists x[\forall y[x \leftrightarrow y]]$

- semantics given as expansion of quantifiers

$$\exists x[f] \;\equiv\; f[0/x] \vee f[1/x] \qquad \forall x[f] \;\equiv\; f[0/x] \wedge f[1/x]$$

- expansion as translation from SAT to QBF is exponential
  - SAT problems have only existential quantifiers
  - expansion of universal quantifiers doubles formula size
- most likely no polynomial translation from SAT to QBF
  - otherwise PSPACE = NP

Tic-Tac-Toe

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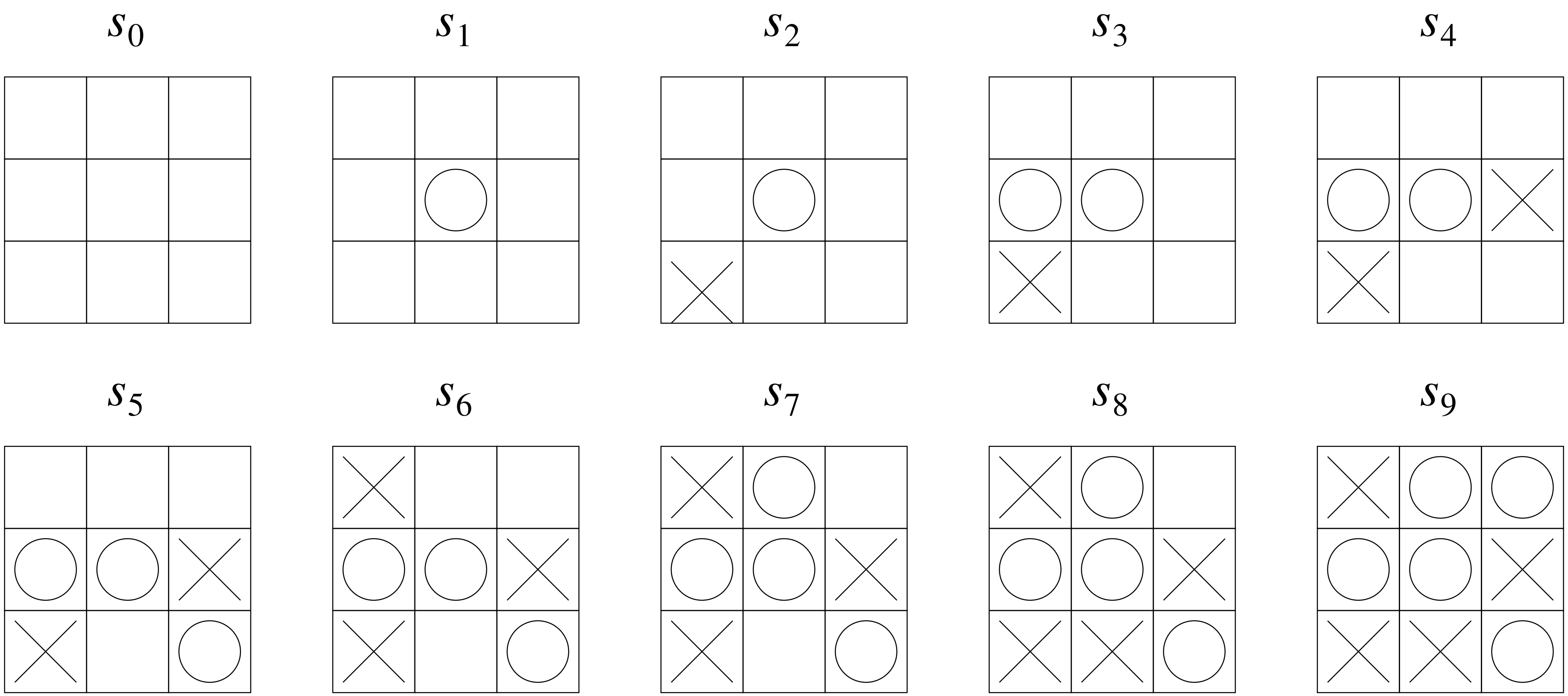

No Winning Strategy for Tic-Tac-Toe

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$$
\begin{array}{l}
\not\models \quad \forall s_0[\mathit{empty}(s_0) \rightarrow \\
\quad \exists x_1[\mathit{circle}(s_0,x_1,s_1) \wedge \\
\quad\quad \forall y_2[\mathit{cross}(s_1,y_2,s_2) \rightarrow \\
\quad\quad\quad \exists x_3[\mathit{circle}(s_2,x_3,s_3) \wedge \\
\quad\quad\quad\quad \forall y_4[\mathit{cross}(s_3,y_4,s_4) \rightarrow \\
\quad\quad\quad\quad\quad \exists x_5[\mathit{circle}(s_4,x_5,s_5) \wedge \\
\quad\quad\quad\quad\quad\quad \forall y_6[\mathit{cross}(s_5,y_6,s_6) \rightarrow \\
\quad\quad\quad\quad\quad\quad\quad \exists x_7[\mathit{circle}(s_6,x_7,s_7) \wedge \\
\quad\quad\quad\quad\quad\quad\quad\quad \forall y_8[\mathit{cross}(s_7,y_8,s_8) \rightarrow \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \exists x_9[\mathit{circle}(s_8,x_9,s_9) \wedge \mathit{win}_{\mathit{circle}}(s_9)]]]]]]]]]
\end{array}
$$

x_i, y_i plays (4 bits each)

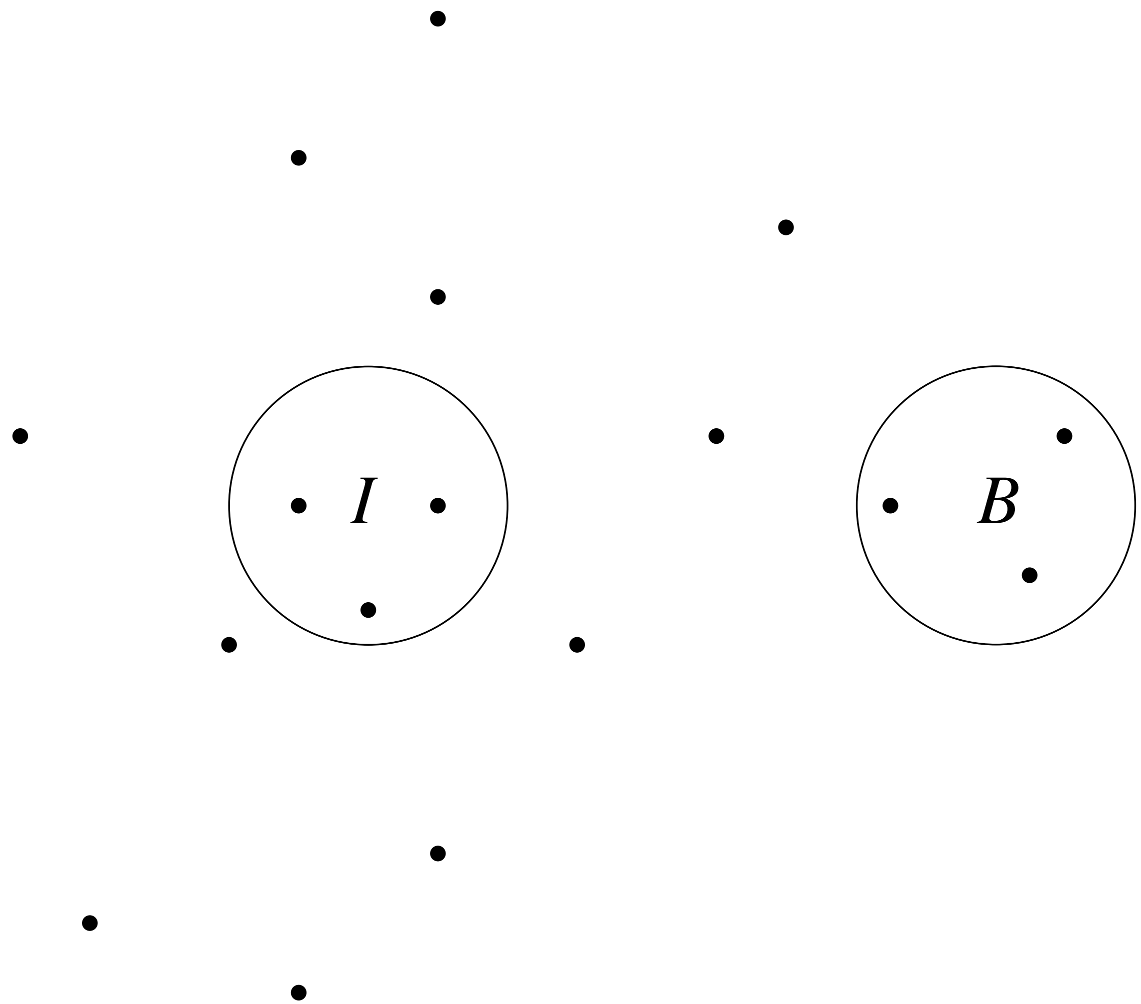
s_i configurations

(9×3 bits each)

- explicit model checking [ClarkeEmerson'82], [Holzmann'91]
  - program presented symbolically (no transition matrix)
  - traversed state space represented explicitly
  - e.g. reached states are explicitly saved bit for bit in hash table

  $\Rightarrow$ State Explosion Problem (state space exponential in program size)

- symbolic model checking [McMillan Thesis'93], [CoudertMadre'89]
  - use symbolic representations for sets of states
  - originally with Binary Decision Diagrams [Bryant'86]
  - Bounded Model Checking using SAT [BiereCimattiClarkeZhu'99]


I
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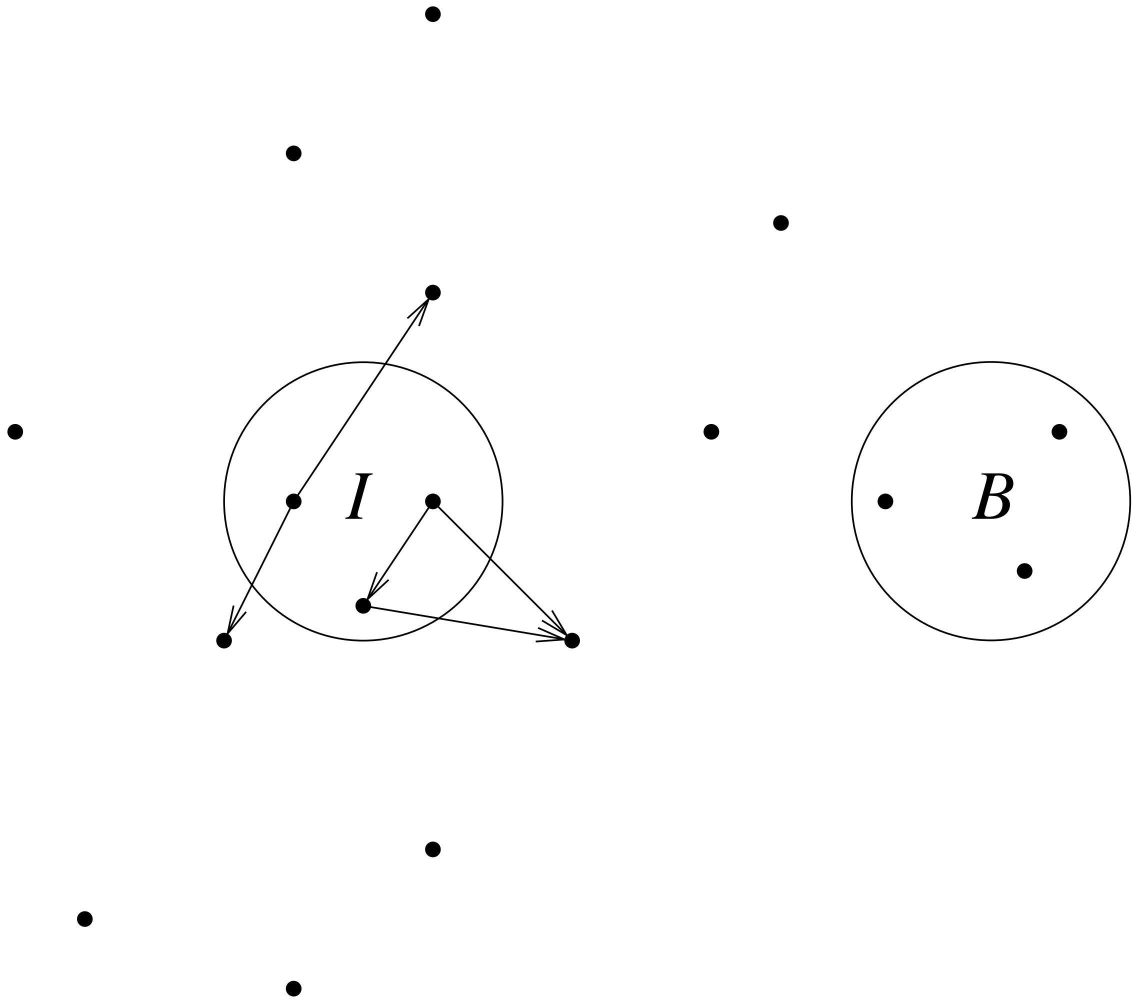
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Forward Fixpoint Algorithm: Step 2

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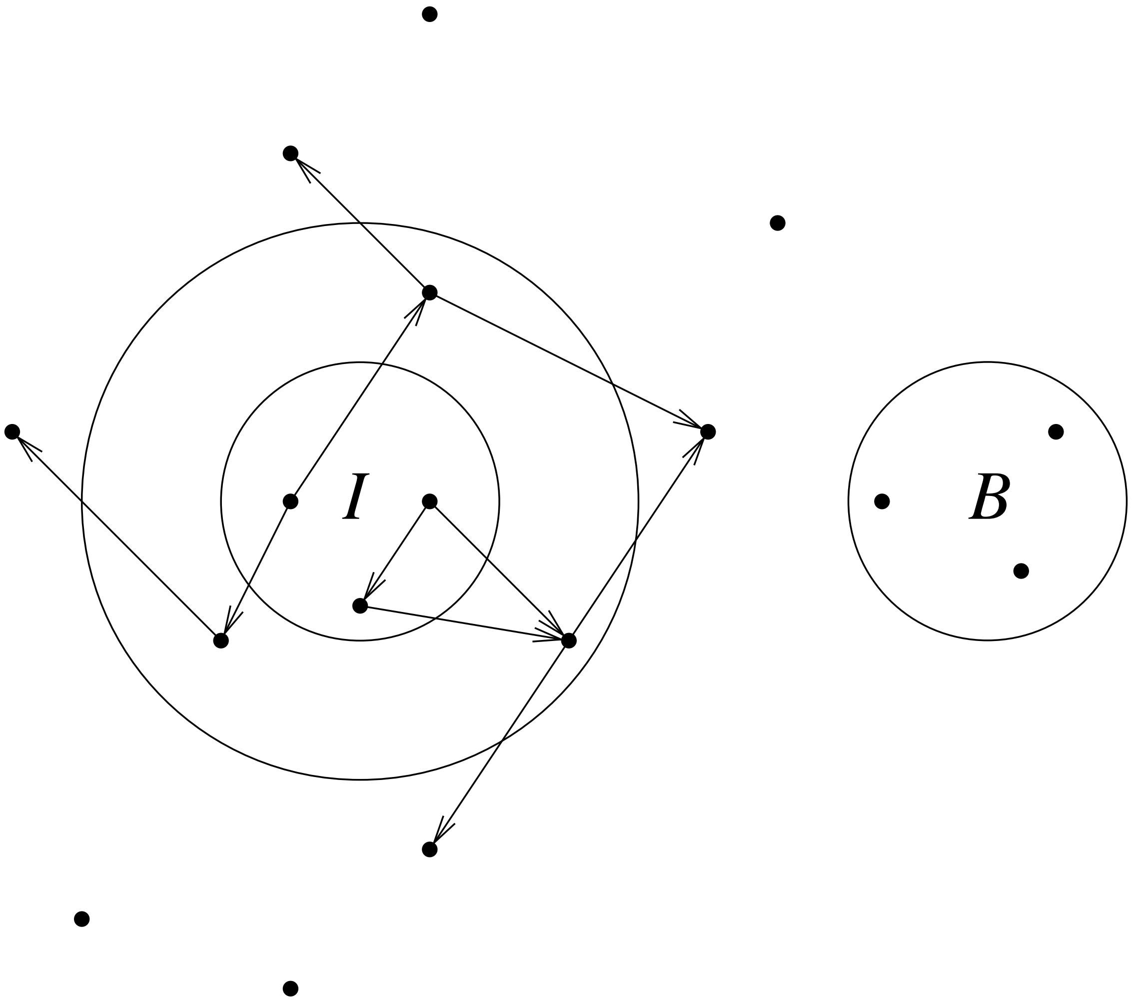
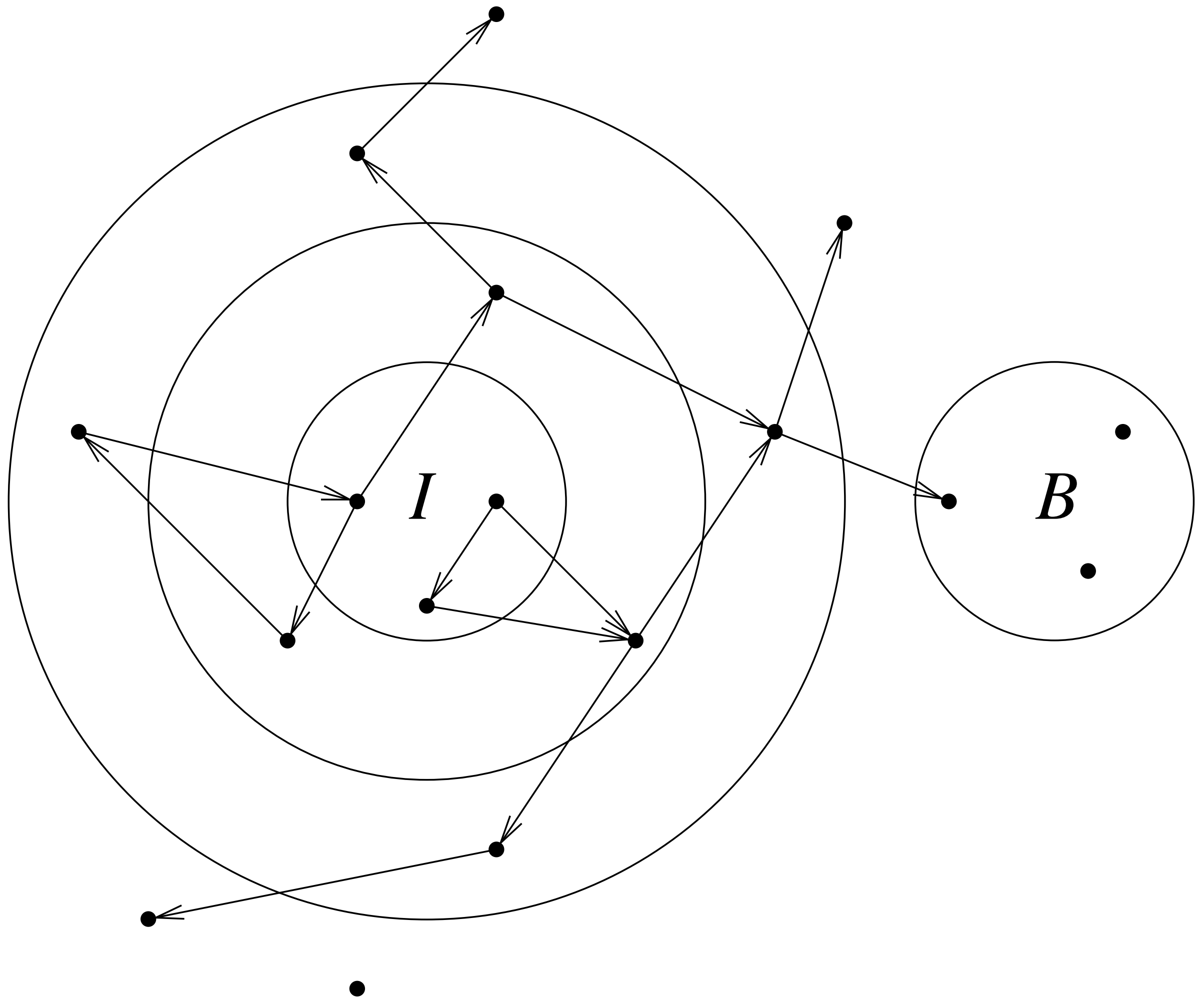
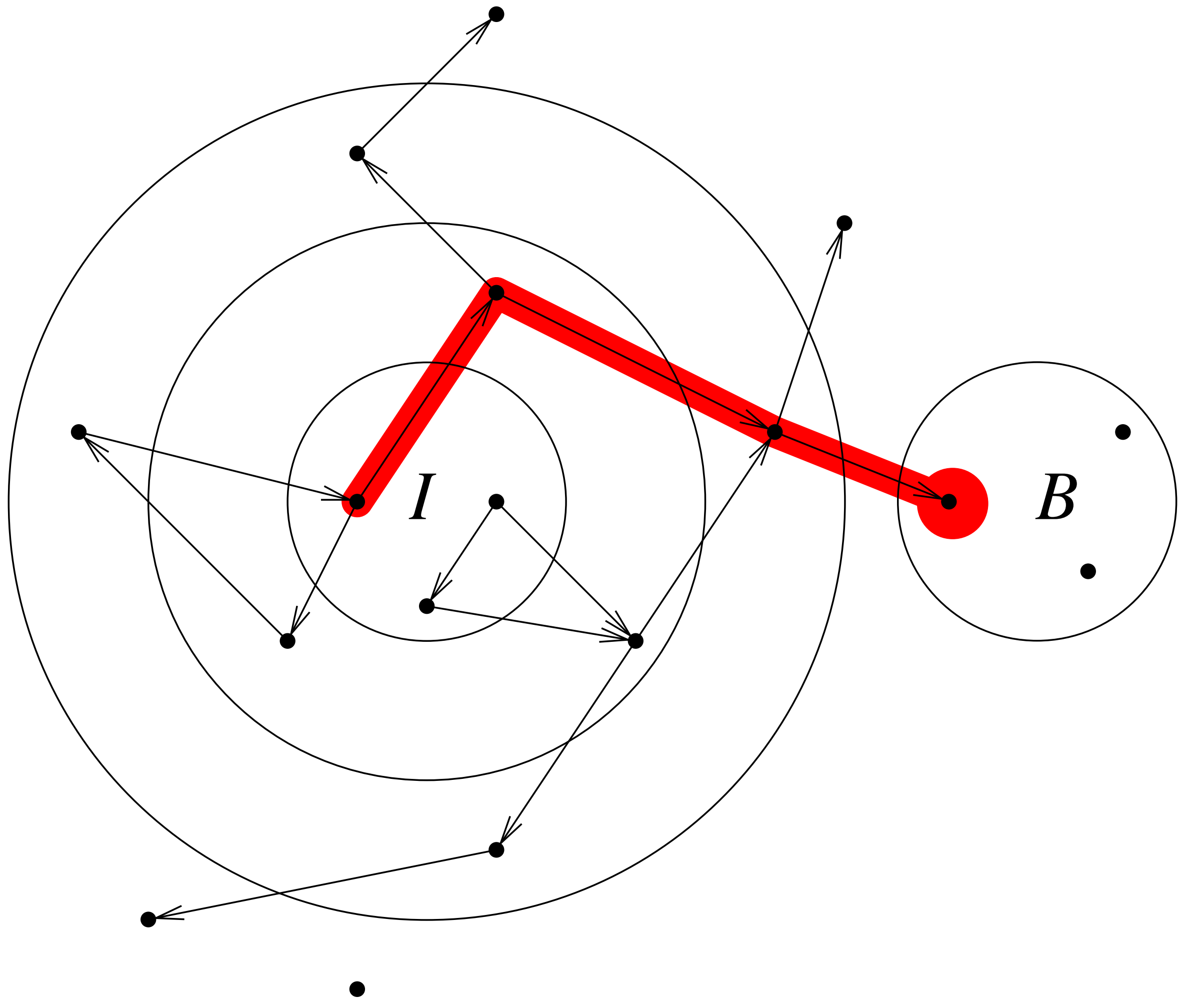


Forward Fixpoint Algorithm: Step 3

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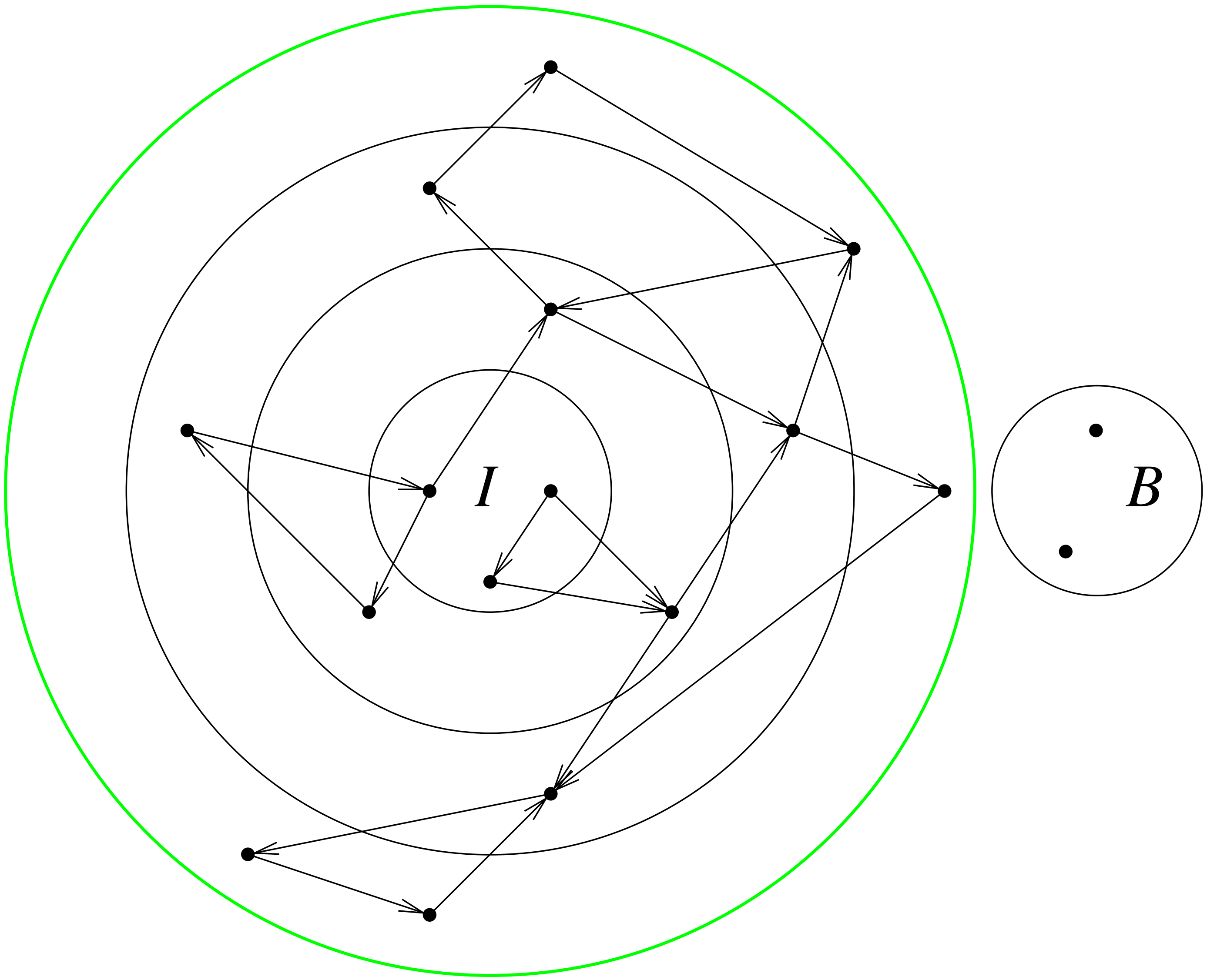
Forward Fixpoint Algorithm: Termination, No Bad State Reachable

A. Biere, FMV, JKU Linz



Forward Least Fixpoint Algorithm for Model Checking Safety

A. Biere, FMV, JKU Linz

initial states I, transition relation T, bad states B

<u>model-check</u>$^{\mu}_{\text{forward}}$ (I, T, B)

$S_C = \emptyset$; $S_N = I$;

while $S_C \neq S_N$ **do**

if $B \cap S_N \neq \emptyset$ **then**

return "found error trace to bad states";

$S_C = S_N$;

$S_N = S_C \cup Img(S_C)$;

done;

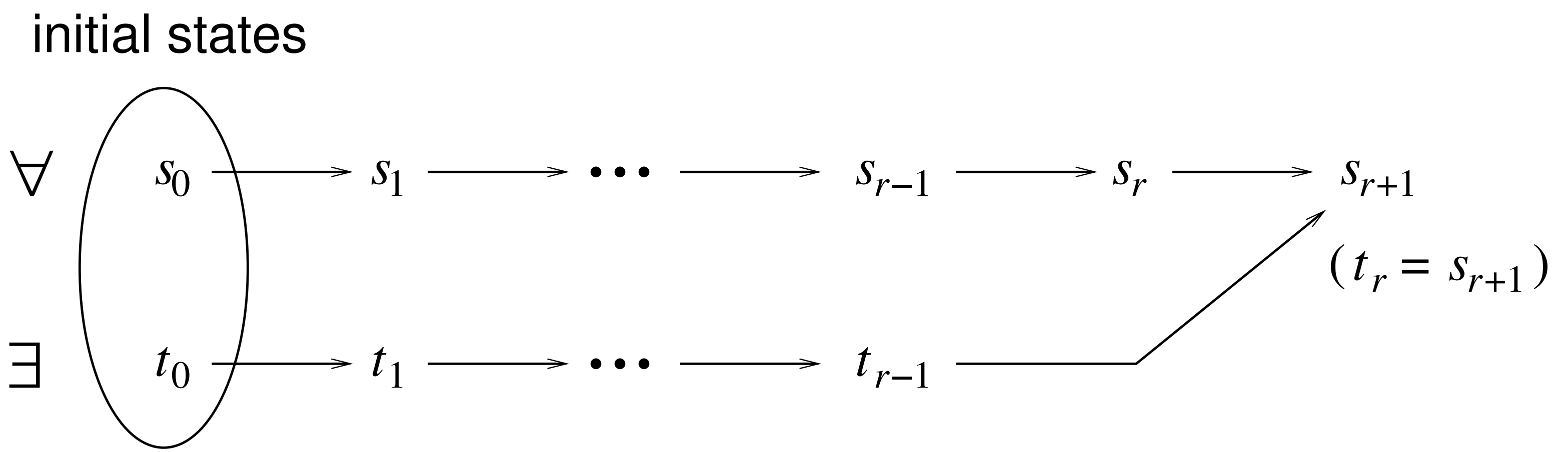
return "no bad state reachable";

symbolic model checking represents set of states in this BFS symbolically

0: continue?	$S_C^0 \neq S_N^0$	$\exists s_0[I(s_0)]$
0: terminate?	$S_C^0 = S_N^0$	$\forall s_0[\neg I(s_0)]$
0: bad state?	$B \cap S_N^0 \neq \emptyset$	$\exists s_0[I(s_0) \wedge B(s_0)]$
1: continue?	$S_C^1 \neq S_N^1$	$\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge \neg I(s_1)]$
1: terminate?	$S_C^1 = S_N^1$	$\forall s_0, s_1[I(s_0) \wedge T(s_0, s_1) \rightarrow I(s_1)]$
1: bad state?	$B \cap S_N^1 \neq \emptyset$	$\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge B(s_1)]$
2: continue?	$S_C^2 \neq S_N^2$	$\exists s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge \neg (I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)])]$
2: terminate?	$S_C^2 = S_N^2$	$\forall s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \rightarrow I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]]$
2: bad state?	$B \cap S_N^1 \neq \emptyset$	$\exists s_0, s_1, s_2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge B(s_2)]$

$$\forall s_0,\ldots,s_{r+1}[\,I(s_0)\wedge\bigwedge_{i=0}^{r}T(s_i,s_{i+1})\rightarrow$$

$$\exists t_0,\ldots,t_r[\,I(t_0)\ \wedge\ s_{r+1}=t_r\ \wedge\ \bigwedge_{i=0}^{r-1}(t_i=t_{i+1}\ \vee\ T(t_i,t_{i+1}))]]$$



(we allow t_{i+1} to be identical to t_i in the lower path)

radius is smallest r for which formula is true

Radius Example

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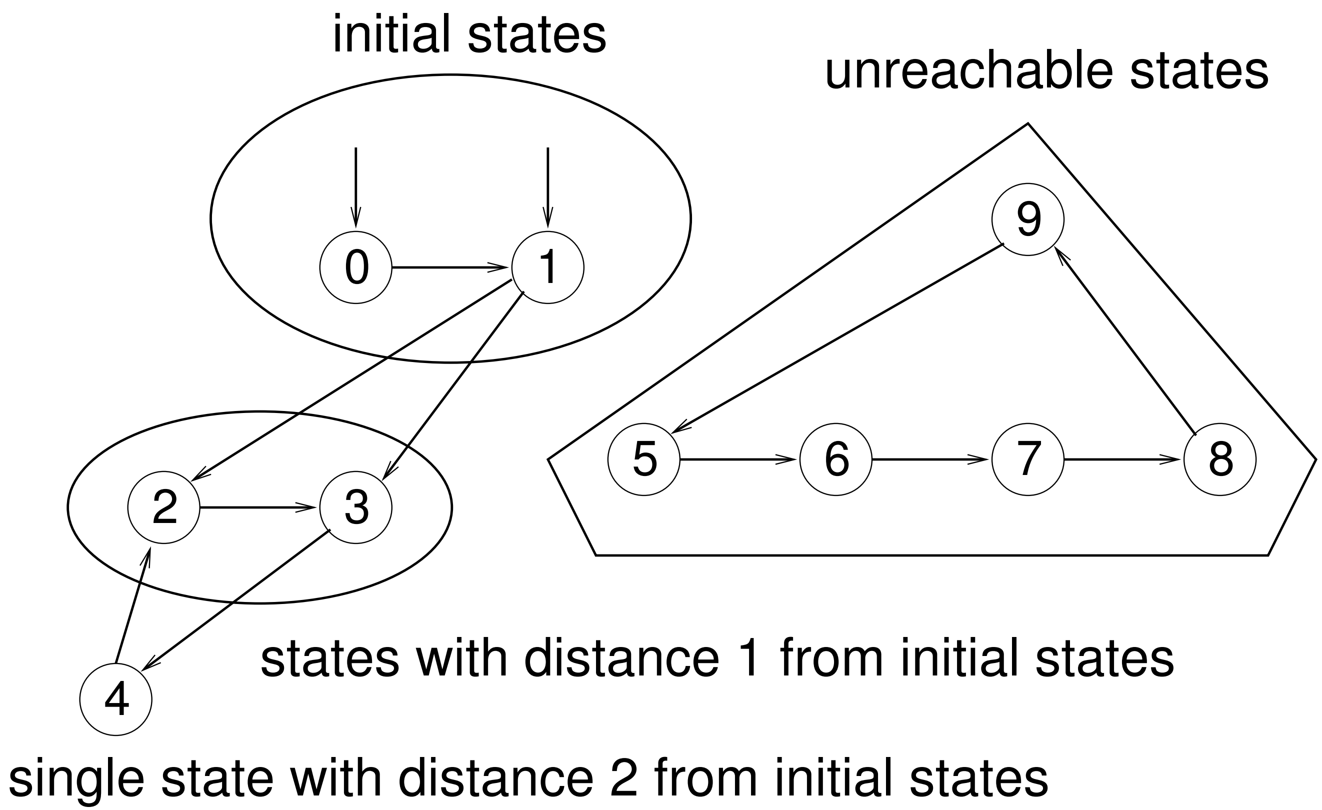

Termination Check in Symbolic Reachability is in QBF

- checking $S_C = S_N$ in 2nd iteration results in QBF decision problem

$$\forall s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \rightarrow I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]]$$

- not **eliminating quantifiers** results in QBF with one alternation
  - checking whether bad state is reached only needs SAT
  - number iterations bounded by radius $r = O(2^n)$
- so why not forget about termination and concentrate on bug finding?

$\Rightarrow$ Bounded Model Checking (BMC)

BMC Part of Fixpoint Algorithm

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~~0: continue?~~ $S_C^0 \neq S_N^0$ $\exists s_0[I(s_0)]$

~~0: terminate?~~ $S_C^0 = S_N^0$ $\forall s_0[\neg I(s_0)]$

0: bad state? $B \cap S_N^0 \neq \emptyset$ $\boxed{\exists s_0[I(s_0) \wedge B(s_0)]}$

~~1: continue?~~ $S_C^1 \neq S_N^1$ $\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge \neg I(s_1)]$

~~1: terminate?~~ $S_C^1 = S_N^1$ $\forall s_0, s_1[I(s_0) \wedge T(s_0, s_1) \rightarrow I(s_1)]$

1: bad state? $B \cap S_N^1 \neq \emptyset$ $\boxed{\exists s_0, s_1[I(s_0) \wedge T(s_0, s_1) \wedge B(s_1)]}$

~~2: continue?~~ $S_C^2 \neq S_N^2$ $\exists s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge$
$\neg(I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]])]$

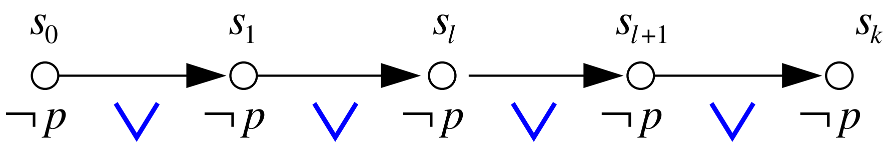
~~2: terminate?~~ $S_C^2 = S_N^2$ $\forall s_0, s_1, s2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \rightarrow$
$I(s_2) \vee \exists t_0[I(t_0) \wedge T(t_0, s_2)]]]$

2: bad state? $B \cap S_N^1 \neq \emptyset$ $\boxed{\exists s_0, s_1, s_2[I(s_0) \wedge T(s_0, s_1) \wedge T(s_1, s_2) \wedge B(s_2)]}$

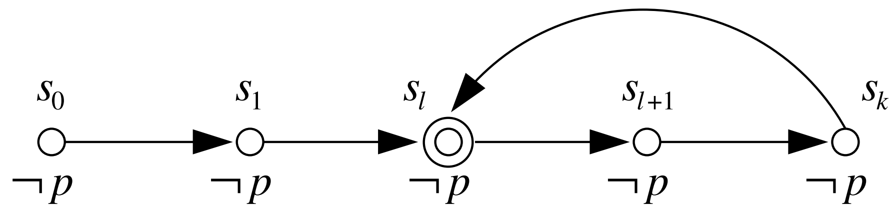
Bounded Model Checking (BMC)

[BiereCimattiClarkeZhu TACAS'99]

- look only for counter example made of k states (the bound)



or



- simple for safety properties $\mathbf{G}p$ (e.g. $p = \neg B$)

$$I(s_0) \wedge \left(\bigwedge_{i=0}^{k-1} T(s_i, s_{i+1})\right) \wedge \bigvee_{i=0}^{k} \neg p(s_i)$$

- harder for liveness properties $\mathbf{F}p$

$$I(s_0) \wedge \left(\bigwedge_{i=0}^{k-1} T(s_i, s_{i+1})\right) \wedge \left(\bigvee_{l=0}^{k} T(s_k, s_l)\right) \wedge \bigwedge_{i=0}^{k} \neg p(s_i)$$

- increase in efficiency of SAT solvers [ZChaff,MiniSAT,SATelite]

- SAT more robust than BDDs in bug finding

  (shallow bugs are easily reached by explicit model checking or testing)

- better unbounded but still SAT based model checking algorithms

  - k-induction [SinghSheeranStålmarck'00]

  - interpolation [McMillan'03]

- 3rd Intl. Workshop on Bounded Model Checking (BMC'05)

- other logics beside LTL and better encodings

  e.g. [LatvalaBiereHeljankoJuntilla'04]

Transitive Closure

$$T^* \quad \equiv \quad T^{2^n}$$

(assuming $\quad = \subseteq T$)

Standard Linear Unfolding

$$T^{i+1}(s,t) \equiv \exists\, m[\, T^i(s,m) \land T(m,t)]$$

Iterative Squaring via Copying

$$T^{2\cdot i}(s,t) \equiv \exists\, m[\, T^i(s,m) \land T^i(m,t)]$$

Non Copying Iterative Squaring

$$T^{2\cdot i}(s,t) \quad \equiv \quad \exists\, m[\forall c[\exists l,r[(c \rightarrow (l,r) = (s,m)) \land (\bar{c} \rightarrow (l,r) = (m,t)) \land T^i(l,r)]]]$$

dpll-sat(*Assignment* S) [DavisLogemannLoveland62]

boolean-constraint-propagation()

if contains-empty-clause() **then return** *false*

if no-clause-left() **then return** *true*

v := next-unassigned-variable()

return dpll-sat(S $\cup \{v \mapsto$ *false*$\}$) $\vee$ dpll-sat(S $\cup \{v \mapsto$ *true*$\}$)

dpll-qbf(*Assignment* S) [CadoliGiovanardiSchaerf98]

boolean-constraint-propagation()

if contains-empty-clause() **then return** *false*

if no-clause-left() **then return** *true*

v := next- outermost -unassigned-variable()

@ := is-existential(v) ? $\vee$: $\wedge$

return dpll-sat(S $\cup \{v \mapsto$ *false*$\}$) @ dpll-sat(S $\cup \{v \mapsto$ *true*$\}$)

Why is QBF harder than SAT?

$$\models \quad \forall x \, . \, \exists y \, . \, (x \leftrightarrow y)$$

$$\not\models \quad \exists y \, . \, \forall x \, . \, (x \leftrightarrow y)$$

Decision order matters!

- most implementations DPLL alike: [Cadoli...98][Rintanen01]
  - **learning** was added [Giunchiglia...01] [Letz01] [ZhangMalik02]
  - top-down: split on variables from the outside to the inside
- multiple quantifier elimination procedures:
  - **enumeration** [PlaistedBiereZhu03] [McMillan02]
  - **expansion** [Aziz-Abdulla...00] [WilliamsBiere...00] [AyariBasin02]
  - bottom-up: eliminate variables from the inside to the outside
- **q-resolution** [KleineBüning...95], with **expansion** [Biere04]
- symbolic representations [PanVardi04] [Benedetti05] BDDs

- applications fuel interest in SAT
  - incredible capacity increase (this year: MiniSAT, SATelite)
  - SAT solver competition affiliated to SAT conference
  - SAT is becoming a core verification technology
- QBF is catching up
  - solvers are getting better
  - new applications
  - richer structure