Offline SMT for Arrays

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Satisfiability Modulo Theories (SMT)

- Generalization of the Boolean Satisfiability Problem (SAT)

- Satisfiability with respect to background theories

- Software and Hardware verification

- SMT Solvers
  - Z3, CVC3, STP, Barcelogic, Boolector, MathSAT, Spear, ...

- Theories
  - Fragments of first-order logic (typically decidable)
  - For example fixed-size Bit-vectors, extensional Arrays
```c
int next_power_of_two (int x)
{
    int i;
    x--;   
    for (i = 1; i < sizeof (int) * 8; i = i * 2)
        x = x | (x >> i);
    return x + 1;
}
```

- next_power_of_two (5) = 8, next_power_of_two (8) = 8, ...

- From the book “Hacker’s Delight” [Warren02]

- Do you trust this algorithm?
Verification instance for bit-width = 4

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Arrays

- Theory of Arrays [McCarthy62]

\((A1)\) \quad a = b \land i = j \Rightarrow read(a, i) = read(b, j)

\((A2)\) \quad i = j \Rightarrow read(write(a, i, e), j) = e

\((A3)\) \quad i \neq j \Rightarrow read(write(a, i, e), j) = read(a, j)

- With \((A1)\) to \((A3)\) we cannot handle array inequalities

- We need additional axiom of extensionality \((A4)\) resp. \((A4')\)

\((A4)\) \quad a = b \iff \forall i (read(a, i) = read(b, i))

\((A4')\) \quad a \neq b \Rightarrow \exists \lambda (read(a, \lambda) \neq read(b, \lambda))
Verification of Selection Sort for bit-width = 32 and size = 4

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• Do not translate the whole formula to SAT

• Let SAT solver “guess” solution
  – If SAT solver cannot find a solution, terminate with \textit{unsatisfiable}

• \textbf{Explicitly check constraints} that were not translated to SAT

• If check succeeds then terminate with \textit{satisfiable}

• If check fails
  – \textbf{Add lemma to refine formula}
  – Let SAT solver “guess” a new solution
• Translate bit-vector but not array part of the formula

• Let SAT solver “guess” solution
  – If SAT solver cannot find a solution, terminate with unsatisfiable

• Explicitly check Array Axioms A1 to A3

• If check succeeds then terminate with satisfiable

• If check fails
  – Add lemma to refine formula
  – Let SAT solver “guess” a new solution
Propagation-based algorithm

- Direct application of (A1) to (A3)

Annotate every array expression with its set of reads $\rho$

For every read $\text{read}(b, i) \in \rho(\text{write}(a, j, e))$

- (A2): If current assignment $\sigma(i) = \sigma(j)$, check if $\sigma(\text{read}(b, i)) = \sigma(e)$
- (A3): If current assignment $\sigma(i) \neq \sigma(j)$, add read to $\rho(a)$

Check read congruence (A1) on all array expressions

Propagation only downwards and can be implemented with DFS or BFS
\( \sigma(i) = \sigma(j), \quad \sigma(r1) \neq \sigma(r2), \quad \sigma(i) \neq \sigma(l1), \quad \sigma(j) \neq \sigma(l2), \quad \sigma(k) = \sigma(l2), \quad \sigma(r3) \neq \sigma(e2) \)
• We have found two inconsistent reads $r_1$ and $r_2$

• We collect all assignments that has been responsible for propagation

• We add a lemma of the following kind:

$$\neg i \neq l_1 \land j \neq l_2 \land \ldots \land i = j \Rightarrow r_1 = r_2$$

• Lemma for inconsistency of $r_1$ and $r_2$ in example 1

$$\neg i \neq l_1 \land j \neq l_2 \land i = j \Rightarrow r_1 = r_2$$

• Lemma for inconsistency of $r_3$ and right *write* in example 1

$$k = l_2 \Rightarrow r_3 = e_2$$
• For every array equality \( a = b \)
  
  – Introduce fresh Tseitin variable \( e_{a,b} \)
  
  – Introduce two virtual reads \( \text{read}(a, \lambda), \text{read}(b, \lambda) \), for a fresh \( \lambda \)
  
  – Virtual reads are used as witness for array inequality
  
  – Encode \( \overline{e}_{a,b} \Rightarrow \text{read}(a, \lambda) \neq \text{read}(b, \lambda) \)

• If \( \sigma(e_{a,b}) = 1 \), propagate reads over array equalities
  
  – Ensures read congruence over equal arrays
  
  – Propagation can now be cyclic, e.g. \( a = b \land b = c \land c = a \)
  
  – We need fix-point algorithm
• We must ensure congruence on write values for equal writes

• For example \( write(a, i, e_1) = write(a, i, e_2) \) implies that \( e_1 = e_2 \)

• We can treat every \( write(a, i, e) \) as \( read(a, i) \), where \( read(a, i) = e \)

• Propagate writes as reads

• In order to reach every array equality
  - We must also propagate upwards with respect to (A2) and (A3)
  - Only propagate upwards if value has not been overwritten
Example 2

\[ \text{write}(a, i, e_1) = \text{write}(b, j, e_2) \land i \neq k \land j \neq k \land \text{read}(a, k) \neq \text{read}(b, k) \]
SMT

Lemmas on demand for **Extensional Theory of Arrays**
- In our examples with Bit-vectors, but approach is more general
- SAT solver is used **offline** as black box

Algorithm based on **propagation** and direct application of array axioms
- Non-extensional algorithm with DFS or BFS
- Introducing equality on arrays requires **fix-point algorithm**
- Virtual reads as witnesses for array **inequalities**

Algorithm implemented in our SMT solver Boolector