What is DQBF?

- DQBF = Dependency Quantified Boolean Formulas

- Boolean formulas with Henkin quantifiers, i.e. dependencies are ...
  - ... specified explicitly
  - ... partially ordered

- Example: $\forall u_1, u_2 \exists e_1(u_1), e_2(u_2).\phi$

- Deciding DQBF is NEXPTIME-complete
Motivation

- Algorithm for solving NEXPTIME-problems (e.g., satisfiability of formulas in EPR or SMT: BV_UF)

- Profit from efficient techniques developed for SAT/QBF

- So far there is no algorithm for DQBF
DQBF $\psi = Q.\phi$, $\phi$ propositional matrix in CNF

$Q$ quantifier prefix of shape:
\[ \forall u_1, \ldots, u_n \exists e_1(u_{1,1}, \ldots, u_{1,k_1}), \ldots, e_m(u_{m,1}, \ldots, u_{m,k_m}), \]
\[ u_{i,j} \in \{u_1, \ldots, u_n\} \]

$\text{dep}(e_i) := \{u_{i,1}, \ldots, u_{i,k_i}\}$ denotes the dependencies of $e_i$

Consider assignments $\beta_1, \beta_2$. Formalization of dependence:
\[ \forall u_j \in \text{dep}(e_i). \beta_1(u_j) = \beta_2(u_j) \rightarrow \beta_1(e_i) = \beta_2(e_i) \]
DQBF $\psi = Q.\phi$, $\phi$ propositional matrix in CNF

$Q$ quantifier prefix of shape:
$$\forall u_1, \ldots, u_n \exists e_1(u_{1,1}, \ldots, u_{1,k_1}), \ldots, e_m(u_{m,1}, \ldots, u_{m,k_m}),$$
$$u_{i,j} \in \{u_1, \ldots, u_n\}$$

$\text{dep}(e_i) := \{u_{i,1}, \ldots, u_{i,k_i}\}$ denotes the dependencies of $e_i$

Consider assignments $\beta_1, \beta_2$. Formalization of dependence:
$$\forall u_j \in \text{dep}(e_i). \beta_1(u_j) = \beta_2(u_j) \rightarrow \beta_1(e_i) = \beta_2(e_i)$$
while (true) do
    state = CheckState($\beta$);
    if (state == UNSAT) then
        HandleConflict();
    else if (state == SAT) then
        HandleSolution();
    else
        literal = SelectLiteral($\beta$);
        AddDecision($\beta$, literal);
    end if
end while
A universal variable \( u_j \) can be picked at any time.

An existential variable \( e_i \) can be selected, if ...

\[
\forall u_j \in \text{dep}(e_i). \beta(u_j) \neq ?
\]

... i.e. all of its dependencies have already been assigned.

More freedom compared to QBF because branching on existential variables can be delayed.
The decision is saved on a decision stack

For an existenial variable $e_i$, $\text{dep}(e_i) = \{u_{i,1}, \ldots, u_{i,k_i}\}$, a Skolem clause $C_{sk}$ is created and linked to the decision

For each decision on the stack the corresponding Skolem clause $C_{sk}$ is considered to be part of the matrix and $F := \phi \land \bigwedge C_{sk}$ has to be satisfied

$C_{sk} = (\overline{l(u_{i,1})} \land \cdots \land \overline{l(u_{i,k_i})} \land l(e_i))$

with $l(x) = \begin{cases} x, & \text{if } \beta(x) = 1 \\ \overline{x}, & \text{if } \beta(x) = 0 \end{cases}$

This forces the algorithm to respect the dependencies
- Look for the last universal that has been picked but the second branch has not been considered yet

- Restore the assignment at the point the universal variable was assigned

- No change to the decision stack occurs during this search
Handle Conflict

- Look for the last existential that has been picked but the second branch has not been considered yet.

- Backtrack and restore the assignment in the same way as it is done in QBF.

- All touched decisions are removed from the stack during this search.

- However, backtracking takes place over trees/several branches because the decision stack was not touched in the SAT-case.
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).\phi \]

\[ \phi = (u_1 \oplus e_2) \land (u_2 \oplus e_1) \]
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).\phi \]

\[ \phi = (u_1 \oplus e_2) \land (u_2 \oplus e_1) \]
\[ = (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2).(u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \]

\[ \beta = (u_1 = ?, u_2 = ?, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

Stack:
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

\[ F = \phi \]

\[ \beta = (u_1 = ?, u_2 = ?, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

Stack:

SelectLiteral: choose from \{u_1, u_2\} \rightarrow u_1 = 0

AddDecision: (u_1 = 0, LB, null)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \]

\[ \beta = (u_1 = 0, u_2 = ?, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (e_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

Stack:

\[
\begin{array}{l}
(u_1 = 0, \text{LB, null})
\end{array}
\]
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

\[ F = \phi \]

\[ \beta = (u_1 = 0, u_2 = ?, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (e_2) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

Stack:

\[
\begin{array}{|c|}
\hline
(u_1 = 0, LB, null) \\
\hline
\end{array}
\]

SelectLiteral: choose from \( \{u_2, e_1\} \rightarrow u_2 = 0 \)

AddDecision: \( (u_2 = 0, LB, null) \)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\bar{u}_1 \lor \bar{e}_2) \land (u_2 \lor e_1) \land (\bar{u}_2 \lor \bar{e}_1) \]

\[ F = \phi \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (e_2) \land (e_1) \]

Stack:

\[
\begin{align*}
(u_2 = 0, \text{LB}, \text{null}) \\
(u_1 = 0, \text{LB}, \text{null})
\end{align*}
\]
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (e_2) \land (e_1) \]

Stack:

- \((u_2 = 0, \text{LB, null})\)
- \((u_1 = 0, \text{LB, null})\)

SelectLiteral: choose from \(\{e_1, e_2\} \rightarrow e_1 = 1\)

AddDecision: \((e_1 = 1, \text{LB, } (u_1 \lor e_1))\)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

\[ F = \phi \land (u_1 \lor e_1) \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = ?) \]

\[ F(\beta) = (e_2) \]

Stack:

- \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
- \((u_2 = 0, \text{LB}, \text{null})\)
- \((u_1 = 0, \text{LB}, \text{null})\)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). \left( (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \right) \]

\[ F = \phi \land \left( u_1 \lor e_1 \right) \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = ?) \]

\[ F(\beta) = (e_2) \]

Stack:

1. \( (e_1 = 1, LB, (u_1 \lor e_1)) \)
2. \( (u_2 = 0, LB, null) \)
3. \( (u_1 = 0, LB, null) \)

SelectLiteral: choose from \( \{ e_2 \} \rightarrow e_2 = 1 \)

AddDecision: \( (e_2 = 1, LB, (u_2 \lor e_2)) \)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = 1) \]

\[ F(\beta) = 1 \]

Stack:

\[
\begin{align*}
(e_2 = 1, LB, (u_2 \lor e_2)) \\
(e_1 = 1, LB, (u_1 \lor e_1)) \\
(u_2 = 0, LB, \text{null}) \\
(u_1 = 0, LB, \text{null})
\end{align*}
\]
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = 1) \]

\[ F(\beta) = 1 \]

Stack:

\[
\begin{align*}
(e_2 = 1, \text{LB}, (u_2 \lor e_2)) \\
(e_1 = 1, \text{LB}, (u_1 \lor e_1)) \\
(u_2 = 0, \text{LB}, \text{null}) \\
(u_1 = 0, \text{LB}, \text{null})
\end{align*}
\]

HandleSolution: find latest universal LB decision

RestoreAssignment: \[ \beta = (u_1 = 0, u_2 = ?, e_1 = ?, e_2 = ?) \]

AddDecision: \( (u_2 = 1, \text{RB}, \text{null}) \)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \]

\[ \beta = (u_1 = 0, u_2 = 1, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (e_2) \land (\overline{e}_1) \land (e_1) \]

Stack:

- \((u_2 = 1, \text{RB}, \text{null})\)
- \((e_2 = 1, \text{LB}, (u_2 \lor e_2))\)
- \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
- \((u_2 = 0, \text{LB}, \text{null})\)
- \((u_1 = 0, \text{LB}, \text{null})\)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \]

\[ \beta = (u_1 = 0, u_2 = 1, e_1 = ?, e_2 = ?) \]

\[ F(\beta) = (e_2) \land (\overline{e_1}) \land (e_1) \]

Stack:

- \((u_2 = 1, \text{RB}, \text{null})\)
- \((e_2 = 1, \text{LB}, (u_2 \lor e_2))\)
- \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
- \((u_2 = 0, \text{LB}, \text{null})\)
- \((u_1 = 0, \text{LB}, \text{null})\)

SelectLiteral: choose from \(\{e_1, e_2\} \rightarrow e_1 = 1\)

AddDecision: \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1)$$

$$F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \land (u_1 \lor e_1)$$

$$\beta = (u_1 = 0, u_2 = 1, e_1 = 1, e_2 = ?)$$

$$F(\beta) = 0$$

Stack:

- $$(e_1 = 1, \text{LB}, (u_1 \lor e_1))$$
- $$(u_2 = 1, \text{RB}, \text{null})$$
- $$(e_2 = 1, \text{LB}, (u_2 \lor e_2))$$
- $$(e_1 = 1, \text{LB}, (u_1 \lor e_1))$$
- $$(u_2 = 0, \text{LB}, \text{null})$$
- $$(u_1 = 0, \text{LB}, \text{null})$$
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \land (u_1 \lor e_1) \]

\[ \beta = (u_1 = 0, u_2 = 1, e_1 = 1, e_2 = ?) \]

\[ F(\beta) = 0 \]

Stack:

- \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
- \((u_2 = 1, \text{RB}, \text{null})\)
- \((e_2 = 1, \text{LB}, (u_2 \lor e_2))\)
- \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
- \((u_2 = 0, \text{LB}, \text{null})\)
- \((u_1 = 0, \text{LB}, \text{null})\)

HandleConflict: backtrack to latest existential LB decision

RestoreAssignment: \( \beta = (u_1 = 0, u_2 = 1, e_1 = ?, e_2 = ?) \)

AddDecision: \((e_1 = 0, \text{RB}, (u_1 \lor \overline{e}_1))\)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \land (u_1 \lor \overline{e_1}) \]

\[ \beta = (u_1 = 0, u_2 = 1, e_1 = 0, e_2 = ?) \]

\[ F(\beta) = 0 \]

Stack:

- \((e_1 = 0, \text{RB}, (u_1 \lor \overline{e_1}))\)
- \((u_2 = 1, \text{RB}, \text{null})\)
- \((e_2 = 1, \text{LB}, (u_2 \lor e_2))\)
- \((e_1 = 1, \text{LB}, (u_1 \lor e_1))\)
- \((u_2 = 0, \text{LB}, \text{null})\)
- \((u_1 = 0, \text{LB}, \text{null})\)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\bar{u}_1 \lor \bar{e}_2) \land (u_2 \lor e_1) \land (\bar{u}_2 \lor \bar{e}_1) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor e_2) \land (u_1 \lor \bar{e}_1) \]

\[ \beta = (u_1 = 0, u_2 = 1, e_1 = 0, e_2 = ?) \]

\[ F(\beta) = 0 \]

Stack:

- \( (e_1 = 1, \text{RB}, (u_1 \lor \bar{e}_1)) \)
- \( (u_2 = 1, \text{RB}, \text{null}) \)
- \( (e_2 = 1, \text{LB}, (u_2 \lor e_2)) \)
- \( (e_1 = 1, \text{LB}, (u_1 \lor e_1)) \)
- \( (u_2 = 0, \text{LB}, \text{null}) \)
- \( (u_1 = 0, \text{LB}, \text{null}) \)

HandleConflict: backtrack to latest existential LB decision

RestoreAssignment: \( \beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = ?) \)

AddDecision: \( (e_2 = 0, \text{RB}, (u_1 \lor \bar{e}_2)) \)
A Simple Example

$$\psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1)$$

$$F = \phi \land (u_1 \lor e_1) \land (u_2 \lor \overline{e}_2)$$

$$\beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = 0)$$

$$F(\beta) = 0$$

Stack:

- $$(e_2 = 0, RB, (u_2 \lor \overline{e}_2))$$
- $$(e_1 = 1, LB, (u_1 \lor e_1))$$
- $$(u_2 = 0, LB, null)$$
- $$(u_1 = 0, LB, null)$$
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u_1} \lor \overline{e_2}) \land (u_2 \lor e_1) \land (\overline{u_2} \lor \overline{e_1}) \]

\[ F = \phi \land (u_1 \lor e_1) \land (u_2 \lor \overline{e_2}) \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = 1, e_2 = 0) \]

\[ F(\beta) = 0 \]

Stack:

- (\( e_2 = 0 \), RB, (\( u_2 \lor \overline{e_2} \))
- (\( e_1 = 1 \), LB, (\( u_1 \lor e_1 \))
- (\( u_2 = 0 \), LB, null)
- (\( u_1 = 0 \), LB, null)

HandleConflict: backtrack to latest existential LB decision

RestoreAssignment: \( \beta = (u_1 = 0, u_2 = 0, e_1 = ?, e_2 = ?) \)

AddDecision: (\( e_1 = 0 \), RB, (\( u_1 \lor \overline{e_1} \)))
ψ = ∀u₁, u₂∃e₁(u₁), e₂(u₂).(u₁ ∨ e₂) ∧ (u₂ ∨ e₁) ∧ (u₂ ∨ e₁) ∧ (u₂ ∨ e₁)
F = φ ∧ (u₁ ∨ e₁)
β = (u₁ = 0, u₂ = 0, e₁ = 0, e₂ = ?)
F(β) = 0

Stack:
(e₁ = 0, RB, (u₁ ∨ e₁))
(u₂ = 0, LB, null)
(u₁ = 0, LB, null)
A Simple Example

\[ \psi = \forall u_1, u_2 \exists e_1(u_1), e_2(u_2). (u_1 \lor e_2) \land (\overline{u}_1 \lor \overline{e}_2) \land (u_2 \lor e_1) \land (\overline{u}_2 \lor \overline{e}_1) \]

\[ F = \phi \land (u_1 \lor \overline{e}_1) \]

\[ \beta = (u_1 = 0, u_2 = 0, e_1 = 0, e_2 = ?) \]

\[ F(\beta) = 0 \]

Stack:

\[
\begin{align*}
(e_1 = 0, \text{RB}, (u_1 \lor \overline{e}_1)) \\
(u_2 = 0, \text{LB}, \text{null}) \\
(u_1 = 0, \text{LB}, \text{null})
\end{align*}
\]

HandleConflict: backtrack to latest existential LB decision

UNSAT
Concepts/Techniques adapted from SAT/QBF

- Unit Propagation
- Pure Literal Reduction
- Clause Learning
- Cube Learning
- Universal Reduction
- Watched Literal Schemes
- Selection Heuristics
Conversion of EPR formulas from the TPTP library to DQBF

Comparison with a QBF solver on QBF benchmarks

Generation of random DQBF instances
Conclusion and Future Work

- First DQBF solver

- DQDPLL architecture based on Skolem clauses
  + Consider expansion based solvers

- Translation for techniques from SAT/QBF
  + Measure single improvements

- Mixed results
  + Optimize and construct more natural benchmarks for DQBF
Questions?