FORMULAS IN CONJUNCTIVE NORMAL FORM (CNF)

VL Logic, Lecture 1, WS 20/21
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Example: Party Planning

We want to plan a party.

Unfortunately, the selection of the guests is not straight forward.

We have to consider the following rules.

1. If two people are married, we have to invite them both or none of them. Alice is married to Bob and Cecile is married to David.

2. If we invite Alice then we also have to invite Cecile. Cecile does not care if we invite Alice but not her.

3. David and Eva can’t stand each other, so it is not possible to invite both.

4. We want to invite Bob and Fred.

Question: Can we find a guest list?
Party Planning with Propositional Logic

- **propositional variables:**
  
  inviteAlice, inviteBob, inviteCecile, inviteDavid, inviteEva, inviteFred

- **constraints:**
  
  1. invite married:  inviteAlice ↔ inviteBob, inviteCecile ↔ inviteDavid
  2. if Alice then Cecile:  inviteAlice → inviteCecile
  3. either David or Eva:  ¬ (inviteEva ↔ inviteDavid)
  4. invite Bob and Fred:  inviteBob ∧ inviteFred

- **encoding in propositional logic:**

  (inviteAlice ↔ inviteBob) ∧ (inviteCecile ↔ inviteDavid) ∧
  (inviteAlice → inviteCecile) ∧ ¬ (inviteEva ↔ inviteDavid) ∧
  inviteBob ∧ inviteFred
Defining a Language: Syntax

What do expressions (words, sentences) of a language look like?

- **set of words**: basic building blocks
- **grammar**: rules for composing sentences from words and smaller sentences
- sometimes multiple (equivalent) representations
  - different goals (user-friendliness vs processability)

- syntactically incorrect formula: 
  \[ \land (a \lor \land b) \land (\neg a \lor \neg b) \]
- syntactically correct propositional formula (over variables \( a \) and \( b \)):
  \[ (a \lor b) \land (\neg a \lor \neg b) \]
- different representation:
  \[ \begin{array}{ccc} 1 & 2 & 0 \\ 3 & 8 \end{array} \]
  \[ \begin{array}{ccc} -1 & -2 & 0 \end{array} \]
Defining a Language: Syntax

What do expressions (words, sentences) of a language look like?

- **set of words**: basic building blocks
- **grammar**: rules for composing sentences from words and smaller sentences
- sometimes multiple (equivalent) representations
  - different goals (user-friendliness vs processability)

example:

- syntactically incorrect formula: $\land(a \lor \land b) \land (\neg a \lor \neg b)$
- syntactically correct propositional formula (over variables $a$ and $b$): $(a \lor b) \land (\neg a \lor \neg b)$
- different representation:
  
  $1 \quad 2 \quad 0$
  
  $-1 \quad -2 \quad 0$
Defining a Language: Semantics

What do expressions mean?

- evaluation of syntactically correct expressions
- logic-based languages have a concise semantics
  - no ambiguities
  - no vagueness

Let propositional variable $a$ be true and propositional variable $b$ be false. Then the truth value of the propositional formula $(a \lor b) \land (\neg a \lor \neg b)$ is true.
Defining a Language: Semantics

What do expressions mean?

- evaluation of syntactically correct expressions
- logic-based languages have a concise semantics
  - no ambiguities
  - no vagueness

Let propositional variable $a$ be true and propositional variable $b$ be false. Then the truth value of the propositional formula

$$(a \lor b) \land (\neg a \lor \neg b)$$

is true.
Using a Language: Pragmatics

What does an expression stand for?

- many application problems can be formulated as logical reasoning problems
- interpretation of expression dependends on context and user
  1. encoding of an application problem to be solved
  2. evaluation of expression gives solution for application problem
Using a Language: Pragmatics

What does an expression stand for?

- many application problems can be formulated as logical reasoning problems
- interpretation of expression dependends on context and user
  1. encoding of an application problem to be solved
  2. evaluation of expression gives solution for application problem

In the formula

\[(a \lor b) \land (\neg a \lor \neg b)\]

\(a\) could stand “invite David” and \(b\) could stand for “invite Eva”. The formula is true if only one of the two persons is invited.
Logical languages allow the inference of new knowledge ("reasoning").

For reasoning, a logic provides various sets of rules (calculi).

Reasoning is often based on certain syntactical patterns.

**example:** (modus ponens)

- $x$ holds.
- If $x$ holds, then also $y$ holds.
- $y$ holds.
Some Remarks on Inferences

A system is inconsistent if we can infer that a statement holds and that a statement does not hold at the same time.

Assume we have modelled the following system

- A comes to the party.
- B comes to the party.
- If A comes to the party, then B does not come to the party.

With the **modus ponens**, we can infer that B does not come to the party.
So, we have some inconsistency in our party model.
Logics in this Lecture

In this lecture, we consider different logic-based languages:

- **part 1: propositional logic (SAT)**
  - simple language: only atoms and connectives
  - low expressiveness, low complexity

- **part 2: first-order logic (predicate logic)**
  - rich language: predicates, functions, terms, quantifiers
  - great power of expressiveness, high complexity

- **part 3: satisfiability modulo theories (SMT)**
  - customizable language: user decides
  - as much expressiveness as required
  - as much complexity as necessary
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SYNTAX OF
PROPOSITIONAL FORMULAS IN CNF

VL Logic
Part I: Propositional Logic
Truth Constants, Variables, Literals

- **truth constant:**
  - `true` (also verum, top): \( \top \)
  - `false` (also falsum, bottom): \( \bot \)
Truth Constants, Variables, Literals

- **truth constant:**
  - *true* (also verum, top): $\top$
  - *false* (also falsum, bottom): $\bot$

- **propositional variable** (also atom, atomic proposition):
  - proposition without any further internal structure
  - we denote variables by symbols $x, y, z, a, b, c, p, q$
    (possibly with subscript)
Truth Constants, Variables, Literals

- **truth constant:**
  - *true* (also verum, top): \( \top \)
  - *false* (also falsum, bottom): \( \bot \)

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    (possibly with subscript)

- **literal:**
  - (negated) truth constant: \( \top, \bot, \neg \top, \neg \bot \)
  - (negated) propositional variable \( x, \neg x, y, \neg y, \ldots \)
  - we denote literals by symbols \( l, k \) (possibly with subscript)
A **clause** is a disjunction ($\lor$) of literals.
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- **examples:** \(x \lor y\) \(x \lor y \lor \neg z\) \(\bot \lor x \lor a \lor T \lor \neg x\)
A clause is a disjunction (\(\lor\)) of literals.

- **Examples:**
  - \(x \lor y\)
  - \(x \lor y \lor \neg z\)
  - \(\bot \lor x \lor a \lor \top \lor \neg x\)

- **Size of a clause:** number of its literals
A **clause** is a disjunction (\( \lor \)) of literals.

- **Examples:**
  - \( x \lor y \)
  - \( x \lor y \lor \neg z \)
  - \( \neg \bot \lor x \lor T \lor \neg x \)

- **Size of a clause:** number of its literals
  - **Empty clause** (size 0): \( \bot \) or \( \emptyset \)
  - **Unary clause** (size 1): \( \neg \bot \lor x \lor \neg z \)
  - **Binary clauses** (size 2): \( x \lor y \)
  - **Ternary clauses** (size 3): \( x \lor y \lor \neg z \)
A clause is a disjunction (∨) of literals.

**Examples:**
- $x \vee y$
- $x \vee y \vee \neg z$
- $\neg \bot \vee x \vee \top \vee \neg x$

**Size of a clause:**
The size of a clause is the number of its literals.

- **Empty clause** (size 0): also written as $\bot$ or $\emptyset$
- **Unary clause** (size 1): $\neg \bot \quad \neg z$
- **Binary clauses** (size 2): $x \vee y$
- **Ternary clauses** (size 3): $x \vee y \vee \neg z$

**Special clauses**
- for $(l_1 \vee \ldots \vee l_n)$ we also write $\bigvee_{i=1}^{n} l_i$
**Clauses**

A clause is a disjunction (\( \lor \)) of literals.

- **Examples:**
  - \( x \lor y \)
  - \( x \lor y \lor \neg z \)
  - \( \neg \bot \lor x \lor \top \lor \neg x \)

- **Size of a clause:** number of its literals
  - **Empty clause** (size 0):
    - (also written as \( \bot \) or \( \emptyset \))
  - **Unary clause** (size 1):
    - \( \neg \bot \lor x \lor \neg z \)
  - **Binary clauses** (size 2):
    - \( x \lor y \)
  - **Ternary clauses** (size 3):
    - \( x \lor y \lor \neg z \)

- **Special clauses**
  - For \( (l_1 \lor \ldots \lor l_n) \) we also write \( \lor_{i=1}^{n} l_i \)

- We denote clauses by symbols \( C, D \) (possibly with subscript)
Propositional Formulas in CNF

A propositional formula in conjunctive normal form is a conjunction ($\land$) of clauses.
Propositional Formulas in CNF

A propositional formula in conjunctive normal form is a conjunction (\(\land\)) of clauses.

- **examples:**
  - \((x \lor T) \land (y \lor \neg z) \land (\neg y \lor \neg x)\)
  - \((\neg x \lor y \lor \neg z) \land z\)
  - \((x \lor \neg y) \land (x \lor \neg y \lor z) \land (y \lor \neg z)\)
  - \(((l_{11} \lor \ldots \lor l_{1m_1}) \land \ldots \land (l_{n1} \lor \ldots \lor l_{nm_n}))\)
A propositional formula in conjunctive normal form is a conjunction (\(\land\)) of clauses.

**Examples:**

- \((x \lor T) \land (y \lor \neg z) \land (\neg y \lor \neg x)\)
- \((\neg x \lor y \lor \neg z) \land z\)
- \((x \lor \neg y) \land (x \lor \neg y \lor z) \land (y \lor \neg z)\)
- \((l_{11} \lor \ldots \lor l_{1m_1}) \land \ldots \land (l_{n1} \lor \ldots \lor l_{nm_n})\)

For \((C_1 \land \ldots \land C_n)\) we also write \(\land_{i=1}^{n} C_i\).
Propositional Formulas in CNF

A propositional formula in conjunctive normal form is a conjunction ($\land$) of clauses.

■ examples:

- $(x \lor T) \land (y \lor \neg z) \land (\neg y \lor \neg x)$
- $(\neg x \lor y \lor \neg z) \land z$
- $(x \lor \neg y) \land (x \lor \neg y \lor z) \land (y \lor \neg z)$
- $((l_{i1} \lor \ldots \lor l_{im_i}) \land \ldots \land (l_{n1} \lor \ldots \lor l_{nm_n}))$

■ for $(C_1 \land \ldots \land C_n)$ we also write $\bigwedge_{i=1}^{n} C_i$.

■ we denote formulas by symbols $\phi, \psi$ (possibly with subscript)
Alternative Notations

in the literature, different symbols might be used for building propositional formulas

<table>
<thead>
<tr>
<th></th>
<th>in this lecture</th>
<th>alternative notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>truth constant “true”</td>
<td>$\top$</td>
<td>1 t true</td>
</tr>
<tr>
<td>truth constant “false”</td>
<td>$\bot$</td>
<td>0 f false</td>
</tr>
<tr>
<td>negation</td>
<td>$\neg x$</td>
<td>!x $\overline{x}$ $\neg x$ NOT $x$</td>
</tr>
<tr>
<td>disjunction</td>
<td>$k \lor l$</td>
<td>$l \mid k$ $l + k$ $l$ OR $k$</td>
</tr>
<tr>
<td>conjunction</td>
<td>$C \land D$</td>
<td>$C$ &amp;&amp; $D$ $C$ * $D$ $C$ AND $D$</td>
</tr>
</tbody>
</table>
We introduced a sublanguage of the language of propositional logic: **propositional formulas in conjunctive normal form (CNF)**

- formula in CNF: conjunction of clauses
- clause: disjunction of literals
- literal: (negated) variable / (negated) truth constant

Any propositional formula can be translated into CNF.
Truth Constants and Variables

- a variable can be assigned one of two values from the two-valued domain $\mathbb{B}$, where $\mathbb{B} = \{1, 0\}$
Truth Constants and Variables

- A variable can be assigned one of two values from the two-valued domain $\mathbb{B}$, where $\mathbb{B} = \{1, 0\}$

- The mapping $\nu : \mathcal{P} \rightarrow \mathbb{B}$ is called assignment, where $\mathcal{P}$ is the set of variables of a formula.

- For $n$ variables, there are $2^n$ assignments possible.

- $\top$ is always true and $\bot$ is always false.
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- The mapping $\nu : \mathcal{P} \rightarrow \mathbb{B}$ is called assignment, where $\mathcal{P}$ is the set of variables of a formula

- We sometimes write an assignment $\nu$ as set $V$ with $V \subseteq \mathcal{P} \cup \{\neg x \mid x \in \mathcal{P}\}$ such that
  - $x \in V$ iff $\nu(x) = 1$
  - $\neg x \in V$ iff $\nu(x) = 0$
Truth Constants and Variables

- A variable can be assigned one of two values from the two-valued domain $B$, where $B = \{1, 0\}$

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  - $\neg x \in V$ iff $\nu(x) = 0$

- for $n$ variables, there are $2^n$ assignments possible

- $\top$ is always true and $\bot$ is always false
Negation Operator

- unary connective ¬ (operator with exactly one operand)
- semantics: flipping the truth value of its operand (under a given assignment)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\neg x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

examples:
- $\neg \bot$ is true.
- $\neg \top$ is false.
- Given assignment $\nu$ and variable $a$ with $\nu(a) = 1$ then $\neg a$ is false under $\nu$.
- Given assignment $\nu$ and variable $a$ with $\nu(a) = 0$ then $\neg a$ is true under $\nu$. 
Binary Disjunction Operator

- binary operator $\lor$ (operator with exactly two operands)
- semantics: true iff at least one operand is true

<table>
<thead>
<tr>
<th>$l$</th>
<th>$k$</th>
<th>$l \lor k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

truth table:

examples:
- $(a \lor \neg a)$ is always true.
- $(\top \lor a)$ is always true.
- The truth value of $(\bot \lor a)$ depends on the truth value of $a$. 
Properties of Disjunction

- **commutative**: \( k \lor l \) is true iff \( l \lor k \) is true, i.e.,

\[ k \lor l \text{ and } l \lor k \text{ are equivalent: } k \lor l \Leftrightarrow l \lor k \]
Properties of Disjunction

- **commutative:** $k \lor l$ is true iff $l \lor k$ is true, i.e.,
  
  $k \lor l$ and $l \lor k$ are equivalent: $k \lor l \Leftrightarrow l \lor k$

- **idempotent:** $l \lor l$ is true iff $l$ is true, i.e.,
  
  $l \lor l$ and $l$ are equivalent: $l \lor l \Leftrightarrow l$
Properties of Disjunction

- **commutative**: \( k \lor l \) is true iff \( l \lor k \) is true, i.e.,
  \[ k \lor l \text{ and } l \lor k \text{ are equivalent: } k \lor l \iff l \lor k \]

- **idempotent**: \( l \lor l \) is true iff \( l \) is true, i.e.,
  \[ l \lor l \text{ and } l \text{ are equivalent: } l \lor l \iff l \]

- **associative**: \( l_1 \lor (l_2 \lor l_3) \) is true iff \( (l_1 \lor l_2) \lor l_3 \) is true, i.e.,
  \[ l_1 \lor (l_2 \lor l_3) \text{ and } (l_1 \lor l_2) \lor l_3 \text{ are equivalent: } l_1 \lor (l_2 \lor l_3) \iff (l_1 \lor l_2) \lor l_3 \]
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- **commutative**: $k \lor l$ is true iff $l \lor k$ is true, i.e.,
  
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  $l_1 \lor (l_2 \lor l_3)$ and $(l_1 \lor l_2) \lor l_3$ are equivalent: $l_1 \lor (l_2 \lor l_3) \iff (l_1 \lor l_2) \lor l_3$

**clauses** are also written as sets

- $(l_1 \lor l_2 \lor \ldots \lor l_{n-1} \lor l_n) = \{l_1, l_2, \ldots, l_{n-1}, l_n\}$

- to add a literal $l$ to clause $C$, we write $C \cup \{l\}$

- to remove a literal $l$ from clause $C$, we write $C \setminus \{l\}$
A clause is true iff at least one of the literals is true.

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>...</th>
<th>$l_n$</th>
<th>$l_1 \lor l_2 \lor \ldots \lor l_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
<td>...</td>
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Truth table:
a clause is true iff at least one of the literals is true

<table>
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<th>$l_1$</th>
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<th>$l_n$</th>
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<td>0</td>
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<td>$\ldots$</td>
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<tr>
<td>1</td>
<td>$\ldots$</td>
<td>1</td>
<td>1</td>
</tr>
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</table>

the empty clause is always false
Binary Conjunction Operator

- binary operator $\land$ (operator with exactly two operands)
- semantics: a conjunction is true iff both operands are true

### Truth Table:

<table>
<thead>
<tr>
<th>$C$</th>
<th>$D$</th>
<th>$C \land D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>1</td>
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<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

### Examples:

- $(a \land \neg a)$ is always false.
- $(\top \land a)$ is true if $a$ is true. $(\bot \land \phi)$ is always false.
- If $(a \lor b)$ is true and $(\neg c \lor d)$ is true then $(a \lor b) \land (\neg c \lor d)$ is true.
Properties of Conjunction

- commutative:
  \[ C \land D \iff D \land C \]

- idempotent:
  \[ C \land C \iff C \]

- associative:
  \[ C_1 \land (C_2 \land C_3) \iff (C_1 \land C_2) \land C_3 \]
Properties of Conjunction

■ commutative:

\[ C \land D \iff D \land C \]

■ idempotent:

\[ C \land C \iff C \]

■ associative:

\[ C_1 \land (C_2 \land C_3) \iff (C_1 \land C_2) \land C_3 \]

Formulas in CNF are also written as sets of sets

■ \(((l_{11} \lor \ldots \lor l_{1m_1}) \land \ldots \land (l_{n1} \lor \ldots \lor l_{nm_n})) = \{\{l_{11}, \ldots l_{1m_1}\}, \ldots, \{l_{n1}, \ldots l_{nm_n}\}\}

■ to add a clause \(C\) to CNF \(\phi\), we write \(\phi \cup \{C\}\)

■ to remove a clause \(C\) from CNF \(\phi\), we write \(\phi \setminus \{C\}\)
CNF Formulas

- A formula in CNF is true iff all of its clauses are true.

Truth table:

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$\ldots$</th>
<th>$C_n$</th>
<th>$C_1 \land C_2 \land \ldots \land C_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$\ldots$</td>
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<td>0</td>
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<td></td>
<td>$\ldots$</td>
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<td>$\ldots$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\ldots$</td>
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<th>$\ldots$</th>
<th>$C_n$</th>
<th>$C_1 \land C_2 \land \ldots \land C_n$</th>
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<td>$\ldots$</td>
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- The empty CNF formula is always true.
Truth Table: Example

the truth table of CNF formula \((x \lor y) \land \neg z\):

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<tr>
<th></th>
<th></th>
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<th>(x \lor y)</th>
<th>(\neg z)</th>
<th>((x \lor y) \land \neg z)</th>
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</table>
Truth Table: Example

the truth table of CNF formula \((x \lor y) \land \neg z\):

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<tr>
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<th></th>
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<th>x \lor y</th>
<th>\neg z</th>
<th>(x \lor y) \land \neg z</th>
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- one assignment: \(\nu(x) = 1, \nu(y) = 0, \nu(z) = 0\)
- alternative notation: \(V = \{x, \neg y, \neg z\}\)
Rules of Precedence

- $\neg$ binds stronger than $\land$
- $\land$ binds stronger than $\lor$
Rules of Precedence

- \( \neg \) binds stronger than \( \land \)
- \( \land \) binds stronger than \( \lor \)

example:

- \( \neg a \lor b \land \neg c \lor d \)
  - is the same as \( (\neg a) \lor (b \land (\neg c)) \lor d \),
  - but not as \( ((\neg a) \lor b) \land ((\neg c) \lor d) \)
Rules of Precedence

- \( \neg \) binds stronger than \( \land \)
- \( \land \) binds stronger than \( \lor \)

Example:

\[ \neg a \lor b \land \neg c \lor d \]

- is the same as \((\neg a) \lor (b \land (\neg c))) \lor d\),
- but not as \(((\neg a) \lor b) \land ((\neg c) \lor d)\)

\[ \Rightarrow \text{ put clauses into parentheses!} \]
Let $\mathcal{P}$ be a set of atoms and $\mathcal{L}$ be the set of all CNFs over $\mathcal{P}$.

Given assignment $\nu : \mathcal{P} \rightarrow \mathbb{B}$, we define the interpretation function $[.]_\nu : \mathcal{L} \rightarrow \mathbb{B}$ by:

- $[\top]_\nu = 1$, $[\bot]_\nu = 0$
- $[x]_\nu = \nu(x)$ (where $x \in \mathcal{P}$, i.e., $x$ is a variable)
- $[\neg p]_\nu = 1$ iff $[p]_\nu = 0$ (where $p \in \mathcal{P} \cup \{\top, \bot\}$)
- $[C]_\nu = 1$ (where $C$ is a clause) iff there is at least one literal $l$ with $l \in C$ and $[l]_\nu = 1$
- $[\phi]_\nu = 1$ (where $\phi$ is in CNF) iff for all clauses $C \in \phi$ it holds that $[C]_\nu = 1$
SAT: THE SATISFIABILITY PROBLEM

VL Logic
Part I: Propositional Logic
Satisfying/Falsifying Assignments

- an assignment $\nu$ is called
  - **satisfying** a formula $\phi$ iff $[\phi]_{\nu} = 1$
  - **falsifying** a formula $\phi$ iff $[\phi]_{\nu} = 0$

- a satisfying assignment for $\phi$ is a **model** of $\phi$
- a falsifying assignment for $\phi$ is a **counter-model** of $\phi$

**example:**

For formula $((x \lor y) \land \neg z)$,

- $\{x, y, z\}$ is a counter-model,
- $\{x, y, \neg z\}$ is a model.
Properties of Propositional Formulas (1/2)

- formula $\phi$ is **satisfiable** iff there exists an assignment $\nu$ with $[\phi]_\nu = 1$
- formula $\phi$ is **valid** iff for all assignments $\nu$ it holds that $[\phi]_\nu = 1$
- formula $\phi$ is **refutable** iff there exists an assignment $\nu$ with $[\phi]_\nu = 0$
- formula $\phi$ is **unsatisfiable** iff for all assignments $\nu$ it holds that $[\phi]_\nu = 0$
Properties of Propositional Formulas (2/2)

- A valid formula is called **tautology**.
- An unsatisfiable formula is called **contradiction**.

**Example:**
- \( \top \) is valid.
- \( a \lor \neg a \) is a tautology.
- \( (a \lor \neg b) \land (\neg a \lor b) \) is refutable.
- \( \bot \) is unsatisfiable.
- \( a \land \neg a \) is a contradiction.
- \( (a \lor \neg b) \land (\neg a \lor b) \) is satisfiable.
SAT: The Boolean Satisfiability Problem

Given a propositional formula $\phi$. Is there an assignment that satisfies $\phi$?
SAT: The Boolean Satisfiability Problem

Given a propositional formula $\phi$. Is there an assignment that satisfies $\phi$?

*formulation for formulas in CNF*: can we find an assignment such that each clause contains at least one true literal?
Satisfiability Checking

- oldest NP-complete problem
  - checking a solution (does the assignment satisfy the formula?) is easy (polynomial effort)
  - finding a solution is difficult (probably exponential in the worst case)

- many practical applications (used in industry)

- efficient SAT solvers (solving tools) are available
VL Logic
Part I: Propositional Logic
SAT-Solver Limboole

- available at http://fmv.jku.at/limboole
- input:\(^1\)
  - variables are strings over letters, digits and \(-\_\.[\[]\]@\(^\)
  - negation symbol \(\neg\) is !
  - disjunction symbol \(\vee\) is |
  - conjunction symbol \(\wedge\) is &

example

\((a \vee b \vee \neg c) \wedge (\neg a \vee b) \wedge c\) is represented as \((a \mid b \mid \neg c) \& (\neg a \mid b) \& c\)

\(^1\)For now, we will only use subset of the language supported by Limboole.
Tool Demo