**Institute for Formal Models and Verification**
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# Propositional Logic

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# Propositions

A proposition is a statement that is either *true or false*.

**Example**

- Alice comes to the party.
- One has to wear a shirt.
- It rains.

With connectives, propositions can be combined to *complex propositions*.

**Example**

- Alice comes to the party and Bob comes to the party, but not Cecile.
- One has to wear either a shirt or a tie.
- If it rains, the street is wet.

# Propositional Logic

- language for representing, combining, and interpreting propositions
- *two truth values (Boolean domain)*: true/false, verum/falsum, on/off, **1**/**0**
- *language elements*
  - atomic propositions (atoms, variables)
    - no internal structure
    - either true or false
  - logic connectives: not ($\neg$), and ($\wedge$), or ($\vee$), ...
    - operators for construction of composite propositions
    - concise meaning
    - argument(s) and return value from Boolean domain
  - parenthesis

## Example

formula of propositional logic: $(\neg \textbf{tie} \vee \textbf{shirt}) \wedge (\textbf{tie} \vee \textbf{shirt}) \wedge (\neg \textbf{tie} \vee \neg \textbf{shirt})$

- atoms: **tie**, **shirt**
- connectives: $\neg$, $\vee$, $\wedge$
- parenthesis for structuring the expression

# Background

- *historical origins*: ancient Greeks
- two very basic principles:
  - *Law of Excluded Middle*: Each expression is either true or false.
  - *Law of Contradiction*: No statement is both true and false.
- very *simple* language
  - no objects, no arguments to propositions
  - no functions
  - no quantifiers
- solving is *easy* (relative to other logics)
- investigated in philosophy, mathematics, and computer science
- propositional logic in computer science:
  - description of digital circuits
  - automated verification
  - planning, scheduling, configuration problems
  - large research area in theoretical computer science
  - many applications in industry

# The Language of Propositional Logic: Syntax

## Definition

The set $\mathcal{L}$ of well-formed propositional formulas is the smallest set such that

1. $\top, \bot \in \mathcal{L}$;
2. $\mathcal{P} \subseteq \mathcal{L}$ where $\mathcal{P}$ is the set of atomic propositions (atoms, variables);
3. if $\phi \in \mathcal{L}$ then $(\neg\phi) \in \mathcal{L}$;
4. if $\phi, \psi \in \mathcal{L}$ then $(\phi \circ \psi) \in \mathcal{L}$ with $\circ \in \{\vee, \wedge, \leftrightarrow, \rightarrow\}$.

$\mathcal{L}$ is the language of propositional logic. The elements of $\mathcal{L}$ are *propositional formulas*.

In *Backus-Naur form (BNF)* propositional formulas are described as follows:

$$\phi ::= \top \mid \bot \mid p \mid (\neg\phi) \mid (\phi \vee \phi) \mid (\phi \wedge \phi) \mid (\phi \leftrightarrow \phi) \mid (\phi \rightarrow \phi)$$

## Example

- $\top$
- $a$
- $(\neg a)$
- $(\neg\top)$
- $(\neg(\neg a))$
- $(a_1 \vee a_2)$
- $(\neg(a \vee b))$
- $(\neg(a \leftrightarrow b))$
- $(((\neg a) \vee a') \leftrightarrow (b \rightarrow c))$
- $(((a_1 \vee a_2) \vee (a_3 \wedge \bot)) \rightarrow b)$

# Rules of Precedence

To reduce the number of parenthesis, we use the following conventions:

- $\neg$ is stronger than $\wedge$
- $\wedge$ is stronger than $\vee$
- $\vee$ is stronger than $\rightarrow$
- $\rightarrow$ is stronger than $\leftrightarrow$
- Binary operators of same strength are assumed to be left parenthesized (also called “left associative”)

**In case of doubt, uses parenthesis!**

## Example

- $\neg a \wedge b \vee c \rightarrow d \leftrightarrow f$ is the same as $(((((\neg a) \wedge b) \vee c) \rightarrow d) \leftrightarrow f)$.
- $a' \vee a'' \vee a'' \wedge b' \vee b''$ is the same as $(((a' \vee a'') \vee (a'' \wedge b')) \vee b'')$.
- $a' \wedge a'' \wedge a'' \vee b' \wedge b''$ is the same as $(((a' \wedge a'') \wedge a''') \vee (b' \wedge b''))$.

# Formula Tree

- formulas have a tree structure
  - *inner nodes*: connectives
  - *leaves*: truth constants, variables

- *default*: inner nodes have one child node (negation) or two nodes as children (other connectives).

- tree structure reflects the use of parenthesis

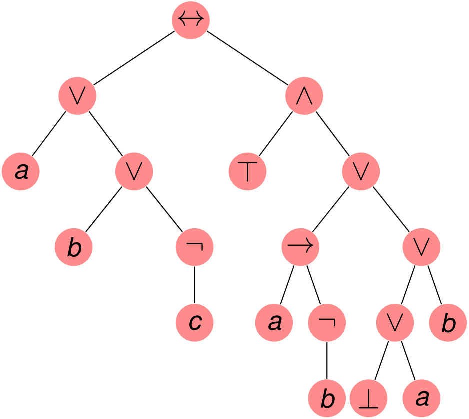
- *simplification*:
  disjunction and conjunction may be considered as *n*-ary operators, i.e., if a node *N* and its child node *C* are of the same kind of connective (conjunction / disjunction), then the children of *C* can become direct children of *N* and the *C* is removed.

# Formula Tree: Example (1/2)

The formula

$$(a \lor (b \lor \neg c)) \leftrightarrow (\top \land ((a \to \neg b) \lor (\bot \lor a \lor b)))$$

has the formula tree

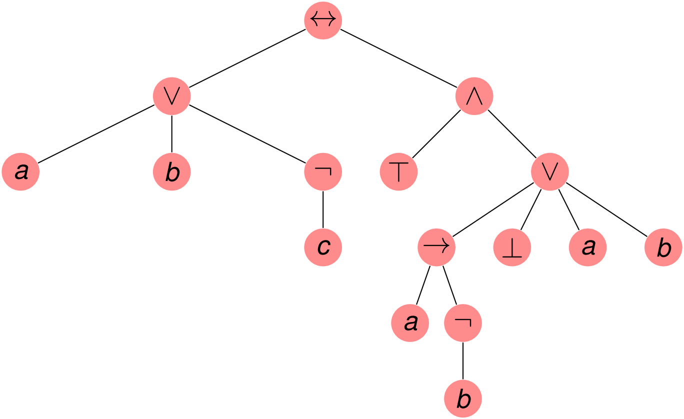


# Formula Tree: Example (2/2)

The formula

$$(a \vee (b \vee \neg c)) \leftrightarrow (\top \wedge ((a \rightarrow \neg b) \vee (\bot \vee a \vee b)))$$

has the simplified formula tree



# Subformulas

## Definition

An *immediate subformula* is defined as follows:

- truth constants and atoms have no immediate subformula.
- only immediate subformula of $\neg\phi$ is $\phi$.
- formula $\phi \circ \psi$ ($\circ \in \{\wedge, \vee, \leftrightarrow, \rightarrow\}$) has immediate subformulas $\phi$ and $\psi$.

The *set of subformulas* of a formula $\phi$ is the smallest set $S$ such that

1. $\phi \in S$
2. if $\psi \in S$ then all immediate subformulas of $\psi$ are in $S$.

*Informal*: A subformula is a part of a formula and is itself a formula.

## Example

The subformulas of $(a \vee b) \rightarrow (c \wedge \neg\neg d)$ are
$\{a, b, c, d, \neg d, \neg\neg d, a \vee b, c \wedge \neg\neg d, (a \vee b) \rightarrow (c \wedge \neg\neg d)\}$

# Limboole

- SAT-solver
- available at `http://fmv.jku.at/limboole/`
- input format:

```
expr    ::= iff
iff     ::= implies { '<->' implies }
implies ::= or [ '->' or | '<-' or ]
or      ::= and { '|' and }
and     ::= not { '&' not }
not     ::= basic | '!' not
basic   ::= var | '(' expr ')'
```

where 'var' is a string over letters, digits, and `- _ . [ ] $ @`

**Example**

In Limboole the formula $(a \vee b) \rightarrow (c \wedge \neg\neg d)$ is represented as

```
((a | b) -> (c & !!d))
```

# Special Formula Structures

- *literal*: variable or a negated variable (also (negated) truth constants)
  - examples of literals: $x, \neg x, y, \neg y$
  - If $l$ is a literal with $l = x$ or $l = \neg x$ then $\text{var}(l) = x$.
  - For literals we use letter $l, k$ (possibly indexed or primed).
  - In principle, we identify $\neg\neg l$ with $l$.

- *clause*: disjunction of literals
  - unary clause (clause of size one): $l$ where $l$ is a literal
  - empty clause (clause of size zero): $\perp$
  - examples of clauses: $(x \vee y)$, $(\neg x \vee x' \vee \neg x'')$, $x, \neg y$

- *cube*: conjunction of literals
  - unary cube (cubes of size one): $l$ where $l$ is a literal
  - empty cubes (cubes of size zero): $\top$
  - examples of cubes: $(x \wedge y)$, $(\neg x \wedge x' \wedge \neg x'')$, $x, \neg y$

# Special Formula Structures: Negation Normal Form

## Definition

*Negation Normal Form* (NNF) is defined as follows:

- Literals and truth constants are in NNF;
- $\phi \circ \psi$ ($\circ \in \{\vee, \wedge\}$) is in NNF iff $\phi$ and $\psi$ are in NNF;
- no other formulas are in NNF.

In other words: A formula in NNF contains only conjunctions, disjunctions, and negations and negations only occur in front of variables and constants.

If a formula is in negation normal form then

- in the formula tree, nodes with negation symbols only occur directly before leaves.
- there are no subformulas of the form $\neg\phi$ where $\phi$ is something else than a variable or a constant.
- it does not contain NAND, NOR, XOR, equivalence, and implication connectives.

## Example

The formula

$((x \vee \neg x_1) \wedge (x \vee (\neg z \vee \neg x_1)))$

is in NNF but

$\neg((x \vee \neg x_1) \wedge (x \vee (\neg z \vee \neg x_1)))$

is not in NNF.

# Special Formula Structures: Conjunctive Normal Form

**Definition**

A propositional formula is in *conjunctive normal form* (CNF) iff it is a conjunction of clauses.

A formula in conjunctive normal form is

- in negation normal form.
- $\top$ if it contains no clauses.
- easy to check whether it can be refuted (can be set to false).

*remark*: CNF is the input of most SAT-solvers (DIMACS format).

**Example**

- $\top$
- $\bot$
- $a$
- $\neg a$
- $l_1 \wedge l_2 \wedge l_3$
- $l_1 \vee l_2 \vee l_3$
- $(a_1 \vee \neg a_2) \wedge (a_1 \vee b_2 \vee a_2) \wedge a_2$
- $((l_{11} \vee \ldots \vee l_{1m_1}) \wedge \ldots \wedge (l_{n1} \vee \ldots \vee l_{nm_n}))$

# Special Formula Structures: Disjunctive Normal Form

## Definition

A propositional formula is in *disjunctive normal form (DNF)* if it is a disjunction of cubes.

A formula in disjunctive normal form is

- in negation normal form.
- $\bot$ if it contains no clauses.
- easy to check whether it can be satisfied (can be set to true).

## Example

- $\top$
- $\bot$
- $a$
- $\neg a$
- $l_1 \wedge l_2 \wedge l_3$
- $l_1 \vee l_2 \vee l_3$
- $(a_1 \wedge \neg a_2) \vee (a_1 \wedge b_2 \wedge a_2) \wedge a_2$
- $((l_{11} \wedge \ldots \wedge l_{1m_1}) \vee \ldots \vee (l_{n1} \wedge \ldots \wedge l_{nm_n}))$

# Conventions

In general, we use the following conventions unless stated otherwise:

- $a, b, c, x, y, z$ denote *variables*.
- $l, k$ denote *literals*.
- $\phi, \psi, \gamma$ denote *arbitrary formulas*.
- $C, D$ denote *clauses or cubes* (clear from context).
- *Clauses* are also written as sets.
  - $(l_1 \vee \ldots \vee l_n) = \{l_1, \ldots l_n\}$.
  - To add a literal $l$ to clause $C$, we write $C \cup \{l\}$.
  - To remove a literal $l$ from clause $C$, we write $C \backslash \{l\}$.
- *Formulas in CNF* are also written as sets of sets.
  - $((l_{11} \vee \ldots \vee l_{1m_1}) \wedge \ldots \wedge (l_{n1} \vee \ldots \vee l_{nm_n})) = \{\{l_{11}, \ldots l_{1m_1}\}, \ldots, \{l_{n1}, \ldots l_{nm_n}\}\}$.
  - To add a clause $C$ to CNF $\phi$, we write $\phi \cup \{C\}$.
  - To remove a clause $C$ from CNF $\phi$, we write $\phi \backslash \{C\}$.

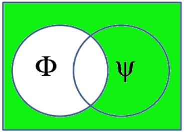
# Elements of Propositional Logic: Negation

- unary connective (operator with exactly one argument)
- negating the truth value of its argument
- alternative notation for $\neg\phi$: $!\phi$, $\overline{\phi}$, $-\phi$, *NOT* $\phi$

*truth table*:

| $\phi$ | $\neg\phi$ |
|---|---|
| **0** | **1** |
| **1** | **0** |

*set view*:




## Example

- If the proposition “It rains.” is true then the negation “It does not rain.” is false.
- If proposition $a$ is true then proposition $\neg a$ is false.
- If formula $((a \vee x) \wedge y)$ is true then formula $\neg((a \vee x) \wedge y)$ is false.
- If proposition $b$ is false, then proposition $\neg b$ is true.
- If formula $((b \rightarrow y) \wedge z)$ is true then formula $\neg((b \rightarrow y) \wedge z)$ is false.

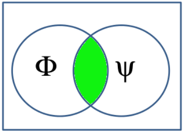
# Elements of Propositional Logic: Conjunction

- a conjunction is true iff both arguments are true
- alternative notation for $\phi \wedge \psi$: $\phi \& \psi$, $\phi\psi, \phi * \psi, \phi \cdot \psi, \phi AND \psi$
- For $(\phi_1 \wedge \ldots \wedge \phi_n)$ we also write $\bigwedge_{i=1}^{n} \phi_i$.
- truth table:

*truth table*:

| $\phi$ | $\psi$ | $\phi \wedge \psi$ |
|---|---|---|
| **0** | **0** | **0** |
| **0** | **1** | **0** |
| **1** | **0** | **0** |
| **1** | **1** | **1** |

*set view*:




## Example

- If the proposition "I want tea." is true and if the proposition "I want cake." is true then also "I want tea and I want cake." is true.
- The proposition $(a \wedge \neg a)$ is false.
- The proposition $(\top \wedge a)$ is true if $a$ is true.
- The proposition $(\bot \wedge a)$ is false.
- If $(a \vee b)$ is true and $(\neg c \vee d)$ is true then $(a \vee b) \wedge (\neg c \vee d)$ is true.

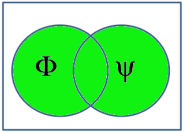
# Elements of Propositional Logic: Disjunction

- a disjunction is true iff at least one of the arguments is true
- alternative notation for $\phi \vee \psi$: $\phi|\psi, \phi + \psi, \phi OR \psi$
- For $(\phi_1 \vee \ldots \vee \phi_n)$ we also write $\bigvee_{i=1}^{n} \phi_i$.

*truth table*:

| $\phi$ | $\psi$ | $\phi \vee \psi$ |
|---|---|---|
| **0** | **0** | **0** |
| **0** | **1** | **1** |
| **1** | **0** | **1** |
| **1** | **1** | **1** |

*set view*:




## Example

- The proposition $(a \vee \neg a)$ is true.
- The proposition $(\top \vee a)$ is true.
- The proposition $(\bot \vee a)$ is true if $a$ is true.
- If $(a \rightarrow b)$ is true and $(\neg c \rightarrow d)$ then $(a \rightarrow b) \vee (\neg c \rightarrow d)$ is true.
- If you see "The menu includes soup or dessert." in a restaurant then this is usually not a disjunction.

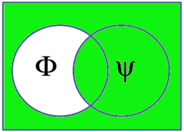
# Elements of Propositional Logic: Implication

- an implication is true iff the first argument is false or both arguments are true
- alternative notation for $\phi \rightarrow \psi$: $\phi \supset \psi$, $\phi \mathit{IMPL} \psi$
- It holds: Verum ex quodlibet. Ex falsum quodlibet.

*truth table*:

| $\phi$ | $\psi$ | $\phi \rightarrow \psi$ |
|---|---|---|
| **0** | **0** | **1** |
| **0** | **1** | **1** |
| **1** | **0** | **0** |
| **1** | **1** | **1** |

*set view*:




## Example

- If the proposition "It rains." is true and the proposition "The street is wet." is true then the statement "If it rains, the street is wet." is true.
- If the proposition "If it rains, the street is wet." is true and the statement "The street is wet." is true, it does not necessarily rain.
- The propositions $(\perp \rightarrow a)$ and $(a \rightarrow a)$ are true.
- The proposition $\top \rightarrow \phi$ is true if $\phi$ is true.

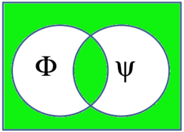
# Elements of Propositional Logic: Equivalence

- binary connective
- an equivalence is true iff both elements have the same value
- alternative notation for $\phi \leftrightarrow \psi$: $\phi = \psi, \phi \equiv \psi, \phi \sim \psi$
- truth table:

*truth table*:

| $\phi$ | $\psi$ | $\phi \leftrightarrow \psi$ |
|---|---|---|
| **0** | **0** | **1** |
| **0** | **1** | **0** |
| **1** | **0** | **0** |
| **1** | **1** | **1** |

*set view*:




## Example

- The formula $a \leftrightarrow a$ is always true.
- The formula $a \leftrightarrow b$ is true iff $a$ is true and $b$ is true or $a$ is false and $b$ is false.
- $\top \leftrightarrow \bot$ is never true.

# Implication vs. Equivalence

In natural language, there is not always a clear distinction between equivalence and implication. The distinction comes from the context.

*equivalence:*

- Iff a student passes a course, (s)he has more than 50 points on the test.
  - To pass the course, it is necessary to have more than 50 points.
  - If a student has more than 50 points on the test then (s)he passes the test.
  - If the student does not have more than 50 points, (s)he does not pass the test.
  - If the student does not pass the test, (s)he did not get at least 50 points.

*implication*:

- If a student passes a course, (s)he has more than 50 points on the test.
  - This statement would also be true if a student fails even though having more than 50 points.
  - Having more than 50 points is necessary, but not sufficient to pass a course.

# The Logic Connectives at a Glance

- The meaning of the connectives can be summarized as follows:

| $\phi$ | $\psi$ | $\top$ | $\bot$ | $\neg\phi$ | $\phi \wedge \psi$ | $\phi \vee \psi$ | $\phi \rightarrow \psi$ | $\phi \leftrightarrow \psi$ | $\phi \oplus \psi$ | $\phi \uparrow \psi$ | $\phi \downarrow \psi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **1** | **0** | **1** | **0** | **0** | **1** | **1** | **0** | **1** | **1** |
| **0** | **1** | **1** | **0** | **1** | **0** | **1** | **1** | **0** | **1** | **1** | **0** |
| **1** | **0** | **1** | **0** | **0** | **0** | **1** | **0** | **0** | **1** | **1** | **0** |
| **1** | **1** | **1** | **0** | **0** | **1** | **1** | **1** | **1** | **0** | **0** | **0** |

## Example

| $\phi$ | $\psi$ | $\neg(\neg\phi \wedge \neg\psi)$ | $\neg\phi \vee \psi$ | $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ |
|---|---|---|---|---|
| **0** | **0** | **0** | **1** | **1** |
| **0** | **1** | **1** | **1** | **0** |
| **1** | **0** | **1** | **0** | **0** |
| **1** | **1** | **1** | **1** | **1** |

*Observation*: connectives can be expressed by other connectives.

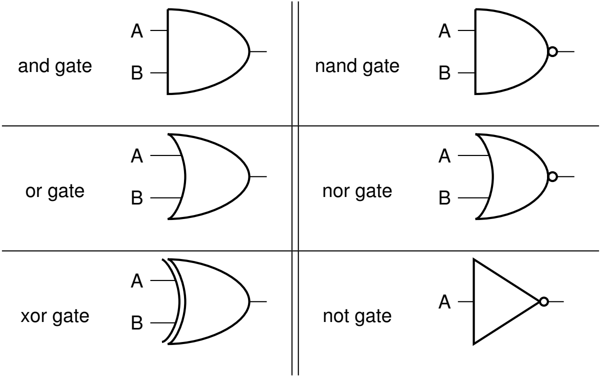
# Other Connectives

- Overall, there are 16 different functions for binary connectives.
- So far, we had conjunction, disjunction, implication, equivalence.
- Further connectives:
  - $\phi \not\leftrightarrow \psi$ (also $\oplus$, *xor*, antivalence)
  - $\phi \uparrow \psi$ (*nand*, Sheffer Stroke Function)
  - $\phi \downarrow \psi$ (*nor*, Pierce Function)

| $\phi$ | $\psi$ | $\phi \not\leftrightarrow \psi$ | $\phi \uparrow \psi$ | $\phi \downarrow \psi$ |
|---|---|---|---|---|
| **0** | **0** | **0** | **1** | **1** |
| **0** | **1** | **1** | **1** | **0** |
| **1** | **0** | **1** | **1** | **0** |
| **1** | **1** | **0** | **0** | **0** |

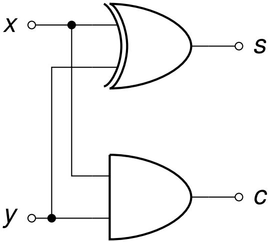
- nor and nand can express every other boolean function (i.e., they are functional complete)
- often used for building digital circuits (like processors)

# Propositional Formulas and Digital Circuits



# Example of a Digital Circuit: Half Adder

| $x$ | $y$ | $c$ | $s$ |
|---|---|---|---|
| **0** | **0** | **0** | **0** |
| **0** | **1** | **0** | **1** |
| **1** | **0** | **0** | **1** |
| **1** | **1** | **1** | **0** |




From the truth table, we see that

$$c \Leftrightarrow x \wedge y$$

and

$$s \Leftrightarrow x \oplus y.$$

# Different Notations

| operator | logic | circuits | Verilog C/C++/Java/C# | VHDL | Limboole |
|---|---|---|---|---|---|
| **1** | $\top$ | 1 | *true* | 1 | – |
| **0** | $\bot$ | 0 | *false* | 0 | – |
| negation | $\neg\phi$ | $\bar{\phi}$   $-\phi$ | $!\phi$ | *not* $\phi$ | $!\,\phi$ |
| conjunction | $\phi \wedge \psi$ | $\phi\psi$   $\phi \cdot \psi$ | $\phi$ && $\psi$ | $\phi$ *and* $\psi$ | $\phi$ & $\psi$ |
| disjunction | $\phi \vee \psi$ | $\phi + \psi$ | $\phi \mid\mid \psi$ | $\phi$ *or* $\psi$ | $\phi \mid \psi$ |
| exclusive or | $\phi \not\leftrightarrow \psi$ | $\phi \oplus \psi$ | $\phi$ != $\psi$ | $\phi$ *xor* $\psi$ | – |
| implication | $\phi \rightarrow \psi$ | $\phi \supset \psi$ | – | – | $\phi$ -> $\psi$ |
| equivalence | $\phi \leftrightarrow \psi$ | $\phi = \psi$ | $\phi$ == $\psi$ | $\phi$ *xnor* $\psi$ | $\phi$ <-> $\psi$ |

## Example

- $(a \vee (b \vee \neg c)) \leftrightarrow (\top \wedge ((a \rightarrow \neg b) \vee (c \vee a \vee b)))$
- $(a + (b + \bar{c})) = c\ ((a \supset -b) + (0 + a + b))$
- $(a \mid\mid (b \mid\mid !c)) == (c\ \&\&\ ((!\ a \mid\mid !\ b) \mid\mid (\mathit{false} \mid\mid a \mid\mid b)))$

# All 16 Binary Functions

| $\phi$ | $\psi$ | constant 0 | nor | | | | | xor | nand | and | equivalence | | implication | | | or | constant 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

# Assignment

- A variable can be assigned one of two values from the two-valued domain $\mathbb{B}$, where $\mathbb{B} = \{\mathbf{1}, \mathbf{0}\}$.
- The mapping $\nu : \mathcal{P} \rightarrow \mathbb{B}$ is called *assignment*, where $\mathcal{P}$ is the set of atomic propositions.
- We sometimes write an assignment $\nu$ as set $V \subseteq \mathcal{P} \cup \{\neg x | x \in \mathcal{P}\}$:
  - $x \in V$ iff $\nu(x) = \mathbf{1}$
  - $\neg x \in V$ iff $\nu(x) = \mathbf{0}$
- For $n$ variables, there are $2^n$ assignments possible.
- An assignment corresponds to one line in the truth table.

## Example

| $x$ | $y$ | $z$ | $(x \vee y) \wedge \neg z$ |
|---|---|---|---|
| **0** | **0** | **0** | **0** |
| **0** | **0** | **1** | **0** |
| **0** | **1** | **0** | **1** |
| **0** | **1** | **1** | **0** |
| **1** | **0** | **0** | **1** |
| **1** | **0** | **1** | **0** |
| **1** | **1** | **0** | **1** |
| **1** | **1** | **1** | **0** |

- One assignment: $\nu(x) = \mathbf{1}, \nu(y) = \mathbf{0}, \nu(z) = \mathbf{1}$
- Alternative notation: $V = \{x, \neg y, z\}$
- *Observation*: A variable assignment determines the truth value of the formulas containing these variables.

# The Language of Propositional Logic: Semantics

## Definition

Given assignment $\nu : \mathcal{P} \rightarrow \mathbb{B}$, the interpretation $[.]_\nu : \mathcal{L} \rightarrow \mathbb{B}$ is defined by:

- $[\top]_\nu = \mathbf{1}$, $[\bot]_\nu = \mathbf{0}$
- if $x \in \mathcal{P}$ then $[x]_\nu = \nu(x)$
- $[\neg\phi]_\nu = \mathbf{1}$ iff $[\phi]_\nu = \mathbf{0}$
- $[\phi \vee \psi]_\nu = \mathbf{1}$ iff $[\phi]_\nu = \mathbf{1}$ or $[\psi]_\nu = \mathbf{1}$

- An assignment is called
  - *satisfying* a formula $\phi$ iff $[\phi]_\nu = \mathbf{1}$.
  - *falsifying* a formula $\phi$ iff $[\phi]_\nu = \mathbf{0}$.
- An assignment satisfying a formula $\phi$ is a *model* of $\phi$.
- An assignment falsifying a formula $\phi$ is a *counter-model* of $\phi$.

## Example

For formula $((x \vee y) \wedge \neg z)$,

- $\{x, y, z\}$ is a counter-model,
- $\{x, y, \neg z\}$ is a model.

# Properties of Propositional Formulas (1/2)

- formula $\phi$ is *satisfiable* iff exists interpretation $[.]_\nu$ with $[\phi]_\nu = \mathbf{1}$
  check with `limboole -s`
- formula $\phi$ is *valid* iff for all interpretations $[.]_\nu$ it holds that $[\phi]_\nu = \mathbf{1}$
  check with `limboole`
- a valid formula is called *tautology*
- formula $\phi$ is *refutable* iff exists interpretation $[.]_\nu$ with $[\phi]_\nu = \mathbf{0}$
  check with `limboole`
- formula $\phi$ is *unsatisfiable* iff $[\phi]_\nu = \mathbf{0}$ for all interpretations $[.]_\nu$
  check with `limboole -s`
- an unsatisfiable formula is called *contradiction*

## Example

- $\top$ is valid.
- $\bot$ is unsatisfiable.
- $(a \lor \neg b) \land (\neg a \lor b)$ is refutable.
- $a \rightarrow b$ is satisfiable.
- $a \leftrightarrow \neg a$ is a contradiction.
- $(a \lor \neg b) \land (\neg a \lor b)$ is satisfiable.

# Properties of Propositional Formulas (2/2)

- A satisfiable formula is
  - possibly valid
  - possibly refutable
  - not unsatisfiable.
- A valid formula is
  - satisfiable
  - not refutable
  - not unsatisfiable.
- A refutable formula is
  - possibly satisfiable
  - possibly unsatisfiable
  - not valid.
- An unsatisfiable formula is
  - refutable
  - not valid
  - not satisfiable.

## Example

- satisfiable, but not valid: $a \leftrightarrow b$
- satisfiable and refutable: $(a \vee b) \wedge (\neg a \vee c)$
- valid, not refutable $\top \vee (a \wedge \neg a)$
- not valid, refutable $(\bot \vee b)$

# Semantic Equivalence

## Definition

Two formula $\phi$ and $\psi$ are *semantic equivalent* (written as $\phi \Leftrightarrow \psi$) iff forall interpretations $[.]_\nu$ it holds that $[\phi]_\nu = [\psi]_\nu$.

Note:

- $\Leftrightarrow$ is a *meta-symbol*, i.e., it is not part of the language.
- *natural language*: if and only if (iff)
- $\phi \Leftrightarrow \psi$ iff $\phi \leftrightarrow \psi$ is valid, i.e., we can express semantics by means of syntactics.
- If $\phi$ and $\psi$ are not equivalent, we write $\phi \not\Leftrightarrow \psi$.

## Example

- $a \vee \neg a \not\Leftrightarrow b \rightarrow \neg b$
- $a \vee \neg a \Leftrightarrow b \vee \neg b$
- $(a \vee b) \wedge \neg(a \vee b) \Leftrightarrow \perp$
- $(a \leftrightarrow (b \leftrightarrow c)) \Leftrightarrow ((a \leftrightarrow b) \leftrightarrow c)$

# Examples of Semantic Equivalences

| | | |
|---|---|---|
| $\phi \wedge \psi \Leftrightarrow \psi \wedge \phi$ | $\phi \vee \psi \Leftrightarrow \psi \vee \phi$ | commutativity |
| $\phi \wedge (\psi \wedge \gamma) \Leftrightarrow (\phi \wedge \psi) \wedge \gamma$ | $\phi \vee (\psi \vee \gamma) \Leftrightarrow (\phi \vee \psi) \vee \gamma$ | associativity |
| $\phi \wedge (\phi \vee \psi) \Leftrightarrow \phi$ | $\phi \vee (\phi \wedge \psi) \Leftrightarrow \phi$ | absorption |
| $\phi \wedge (\psi \vee \gamma) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \gamma)$ | $\phi \vee (\psi \wedge \gamma) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \gamma)$ | distributivity |
| $\neg(\phi \wedge \psi) \Leftrightarrow \neg\phi \vee \neg\psi$ | $\neg(\phi \vee \psi) \Leftrightarrow \neg\phi \wedge \neg\psi$ | laws of De Morgan |
| $\phi \leftrightarrow \psi \Leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ | $\phi \leftrightarrow \psi \Leftrightarrow (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$ | synt. equivalence |
| $\phi \vee \psi \Leftrightarrow \neg\phi \rightarrow \psi$ | $\phi \rightarrow \psi \Leftrightarrow \neg\psi \rightarrow \neg\phi$ | implications |
| $\phi \wedge \neg\phi \Leftrightarrow \bot$ | $\phi \vee \neg\phi \Leftrightarrow \top$ | complement |
| $\neg\neg\phi \Leftrightarrow \phi$ | | double negation |
| $\phi \wedge \top \Leftrightarrow \phi$ | $\phi \vee \bot \Leftrightarrow \phi$ | neutrality |
| $\phi \vee \top \Leftrightarrow \top$ | $\phi \wedge \bot \Leftrightarrow \bot$ | |
| $\neg\top \Leftrightarrow \bot$ | $\neg\bot \Leftrightarrow \top$ | |

# Further Connections between Formulas

- A formula $\phi$ is valid iff $\neg\phi$ is unsatisfiable.
- A formula $\phi$ is satisfiable iff $\neg\phi$ is not valid.
- The formulas $\phi$ and $\psi$ are equivalent iff $\phi \leftrightarrow \psi$ is valid.
- The formulas $\phi$ and $\psi$ are equivalent iff $\neg(\phi \leftrightarrow \psi)$ is unsatisfiable.
- A formula $\phi$ is satisfiable iff $\phi \not\leftrightarrow \bot$.

# Logic Entailment

## Definition

Let $\phi_1, \dots \phi_n, \psi$ be propositional formulas. Then $\phi_1, \dots \phi_n$ *entail* $\psi$ (written as $\phi_1, \dots, \phi_n \models \psi$) iff $[\phi_1]_\nu = \mathbf{1}, \dots [\phi_n]_\nu = \mathbf{1}$ implies that $[\psi]_\nu = \mathbf{1}$.

Informal meaning: True premises derive a true conclusion.

- $\models$ is a *meta-symbol*, i.e., it is not part of the language.
- $\phi_1, \dots \phi_n \models \psi$ iff $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is valid, i.e., we can express semantics by means of syntactics.
- If $\phi_1, \dots \phi_n$ do not entail $\psi$, we write $\phi_1, \dots \phi_n \not\models \psi$.

## Example

- $a \models a \vee b$
- $a, b \models a \wedge b$
- $a, a \rightarrow b \models b$
- $\models a \vee \neg a$
- $\not\models a \wedge \neg a$
- $\bot \models a \wedge \neg a$

# Satisfiability Equivalence

## Definition

Two formulas $\phi$ and $\psi$ are *satisfiability-equivalent* (written as $\phi \Leftrightarrow_{SAT} \psi$) iff both formulas are satisfiable or both are contradictory.

- Satisfiability-equivalent formulas are not necessarily satisfied by the same assignments.
- Satisfiability equivalence is a weaker property than equivalence.
- Often sufficient for simplification rules: If the complicated formula is satisfiable then also the simplified formula is satisfiable.

## Example

*Positive pure literal elimination rule*: If a variable $x$ occurs in a formula but $\neg x$ does not occur in the formula, then $x$ can be substituted by $\top$. The resulting formula is satisfiability-equivalent.

- $x \Leftrightarrow_{SAT} \top$, but $x \not\Leftrightarrow \top$
- $(a \wedge b) \vee (\neg c \wedge a) \Leftrightarrow_{SAT} b \vee \neg c$, but $(a \wedge b) \vee (\neg c \wedge a) \not\Leftrightarrow b \vee \neg c$

# Representing Functions as CNFs

- *Problem*: Given the truth table of a Boolean function $\phi$. How is the function represented in propositional logic?

*Solution (Representation as CNF)*:

1. Represent each assignment $\nu$ where $\phi$ has value **0** as clause:
   - If variable $x$ is **1** in $\nu$, add $\neg x$ to clause.
   - If variable $x$ is **0** in $\nu$, add $x$ to clause.
2. Connect all clauses by conjunction.

**Example**

| a | b | c | $\phi$ | clauses |
|---|---|---|---|---|
| **0** | **0** | **0** | **0** | $a \vee b \vee c$ |
| **0** | **0** | **1** | **1** | |
| **0** | **1** | **0** | **1** | |
| **0** | **1** | **1** | **0** | $a \vee \neg b \vee \neg c$ |
| **1** | **0** | **0** | **1** | |
| **1** | **0** | **1** | **0** | $\neg a \vee b \vee \neg c$ |
| **1** | **1** | **0** | **0** | $\neg a \vee \neg b \vee c$ |
| **1** | **1** | **1** | **1** | |

$$\phi = (a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee c)$$

# Representing Functions as DNFs

- *Problem*: Given the truth table of a Boolean function $\phi$. How is the function represented in propositional logic?

*Solution (Representation as DNF)*:

1. Represent each assignment $\nu$ where $\phi$ has value **1** as cube:
   - If variable $x$ is **1** in $\nu$, add $x$ to cube.
   - If variable $x$ is **0** in $\nu$, add $\neg x$ to cube.
2. Connect all cubes by disjunction.

**Example**

| $a$ | $b$ | $c$ | $\phi$ | cubes |
|---|---|---|---|---|
| **0** | **0** | **0** | **0** | |
| **0** | **0** | **1** | **1** | $\neg a \wedge \neg b \wedge c$ |
| **0** | **1** | **0** | **1** | $\neg a \wedge b \wedge \neg c$ |
| **0** | **1** | **1** | **0** | |
| **1** | **0** | **0** | **1** | $a \wedge \neg b \wedge \neg c$ |
| **1** | **0** | **1** | **0** | |
| **1** | **1** | **0** | **0** | |
| **1** | **1** | **1** | **1** | $a \wedge b \wedge c$ |

$$\phi = (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c)$$

# Functional Completeness

- In propositional logic there are
  - 2 functions of arity 0 ($\top$, $\bot$)
  - 4 functions of arity 1 (e.g., not)
  - 16 functions of arity 2 (e.g., and, or, ...)
  - $2^{2^n}$ functions of arity $n$.
- A function of arity $n$ has $2^n$ different combinations of arguments (lines in the truth table).
- A functions maps its arguments either to **1** or **0**.

## Definition

A set of functions is called *functional complete* for propositional logic iff it is possible to express all other functions of propositional logic with functions from this set.

## Example

- $\{\neg, \wedge\}$ is functional complete.
- $\{\neg, \vee\}$ is functional complete.
- nand is functional complete.
- nor is functional complete.

# Encoding the k-Coloring Problem

Given graph $(V, E)$ with vertices $V$ and edges $E$. Color each node with one of $k$ colors, such that there is no edge $(v, w) \in E$, with vertices $v$ and $w$ colored in the same color.

Encoding:

1. *Propositional variables*: $v_j$ ... node $v \in V$ has color $j$ $(1 \leq j \leq k)$
2. each node has *a color*:
$$\bigwedge_{v \in V} \left( \bigvee_{1 \leq j \leq k} v_j \right)$$
3. each node has *just one color*: $\neg(v_i \wedge v_j)$ with $v \in V, 1 \leq i < j \leq k$
4. neighbors have *different colors*: $\neg(v_i \wedge w_i)$ with $(v, w) \in E, 1 \leq i \leq k$

**Example**

2-coloring of $(\{a, b, c\}, \{(a, b), (b, c)\})$

1. $a_1, a_2, b_1, b_2, c_1, c_2$
2. $a_1 \vee a_2, b_1 \vee b_2, c_1 \vee c_2$
3. $\neg(a_1 \wedge a_2), \neg(b_1 \wedge b_2), \neg(c_1 \wedge c_2)$
4. $\neg(a_1 \wedge b_1), \neg(a_2 \wedge b_2)$
   $\neg(b_1 \wedge c_1), \neg(b_2 \wedge c_2)$

# A Puzzle

> *A lady is in one of two rooms called A and B. A tiger is also in A or B. On the door of A there is a sign: "This room contains a lady, the other room contains a tiger." The door of room B has a sign: "The tiger and the lady are not in the same room." One sign lies. Where is the lady, where is the tiger?*
>
> based on a puzzle by Raymond Smullyan

*One possible SAT encoding:*

- *signOnA* represents that sign of room *A* says the truth
- *signOnB* represents that sign of room *B* says the truth
- **ladyInA** or **ladyInB** represents that lady is in *A* or *B* respectively
- **tigerInA** or **tigerInB** represents that tiger is in *A* or *B* respectively
- lady is in room *A* or *B*, but not in both: $(\mathbf{ladyInA} \vee \mathbf{ladyInB}) \wedge \neg(\mathbf{ladyInA} \wedge \mathbf{ladyInB})$
- tiger is in room *A* or *B*, but not in both: $(\mathbf{tigerInA} \vee \mathbf{tigerInB}) \wedge \neg(\mathbf{tigerInA} \wedge \mathbf{tigerInB})$
- one sign lies, one sign is true: $(\mathit{signOnA} \leftrightarrow \neg \mathit{signOnB})$
- sign of room *A*: $\mathit{signOnA} \leftrightarrow (\mathbf{ladyInA} \wedge \mathbf{tigerInB})$
- sign of room *B*: $\mathit{signOnB} \leftrightarrow (\neg(\mathbf{tigerInA} \wedge \mathbf{ladyInA}) \wedge \neg(\mathbf{tigerInB} \wedge \mathbf{ladyInB}))$