TRANSFORMATION TO CNF

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Transformation to Conjunctive Normal Form 1

Approach 1: Transformation by “multiplication”

1. Remove $\leftrightarrow, \rightarrow, \oplus$ as follows:
   - $\phi \leftrightarrow \psi \Leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$
   - $\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi$,
   - $\phi \oplus \psi \Leftrightarrow (\phi \vee \psi) \wedge (\neg\phi \vee \neg\psi)$
2. Transform formula to negation normal form (NNF) by
   - application of laws of De Morgan
   - elimination of double negation
3. Transform formula to CNF by laws of distributivity

Example: Transformation to CNF 1

Transformation of $\neg(a \leftrightarrow b) \rightarrow (\neg(c \wedge d) \wedge e)$ to an equivalent formula in CNF by approach 1

1. a) remove equivalences:
   $\Leftrightarrow \neg((a \rightarrow b) \wedge (b \rightarrow a)) \rightarrow (\neg(c \wedge d) \wedge e)$
   b) remove implications:
   $\Leftrightarrow \neg\neg((\neg a \vee b) \wedge (\neg b \vee a)) \vee (\neg(c \wedge d) \wedge e)$
2. NNF: $\Leftrightarrow ((\neg a \vee b) \wedge (\neg b \vee a)) \vee ((\neg c \vee \neg d) \wedge e)$
3. $\Leftrightarrow ((\neg a \vee b) \vee ((\neg c \vee \neg d) \wedge e))) \wedge ((\neg b \vee a) \vee ((\neg c \vee \neg d) \wedge e)))$
   $\Leftrightarrow (\neg a \vee b \vee \neg c \vee \neg d) \wedge (\neg a \vee b \vee e) \wedge (\neg b \vee a \vee \neg c \vee \neg d) \wedge (\neg b \vee a \vee e)$

Transformation to Conjunctive Normal Form 2

Approach 2: Transformation by introducing labels for subformulas

Given formula ϕ. The following approach transforms ϕ to an equi-satisfiable formula in CNF.

1. Introduce new label v_ψ for each subformula ψ that is not a literal
2. Collect all definitions $l_\psi \leftrightarrow \psi'$ in a big conjunction Φ' (ψ' is obtained from ψ by replacing its immediate subformulas by the respective labels)
3. Transform Φ' to CNF Φ by approach 1 (no exponential blowup!)

$(\Phi \wedge l_\phi)$ **and** ϕ **are equi-satisfiable**

Example: Transformation to CNF 2

Transform $\phi := \neg(a \leftrightarrow b) \rightarrow (\neg(c \wedge d) \wedge e)$ to an equi-satisfiable formula in CNF

definitions Φ'	clauses Φ
$v_1 \leftrightarrow (a \leftrightarrow b)$	$(\bar{v}_1 \vee \bar{a} \vee b), (\bar{v}_1 \vee a \vee \bar{b}), (v_1 \vee \bar{a} \vee \bar{b}), (v_1 \vee a \vee b)$
$v_2 \leftrightarrow \neg v_1$	$(v_1 \vee v_2), (\bar{v}_1 \vee \bar{v}_2)$
$v_3 \leftrightarrow (c \wedge d)$	$(\bar{v}_3 \vee c), (\bar{v}_3 \vee d), (v_3 \vee \bar{c} \vee \bar{d})$
$v_4 \leftrightarrow \neg v_3$	$(v_3 \vee v_4), (\bar{v}_3 \vee \bar{v}_4)$
$v_5 \leftrightarrow (v_4 \wedge e)$	$(\bar{v}_5 \vee v_4), (\bar{v}_5 \vee e), (v_5 \vee \bar{v}_4 \vee \bar{e})$
$v_6 \leftrightarrow (v_2 \rightarrow v_5)$	$(\bar{v}_6 \vee \bar{v}_2 \vee v_5), (v_2 \vee v_6), (\bar{v}_5 \vee v_6)$

$(\Phi \wedge v_6)$ **and** ϕ **are equi-satisfiable.**

Some Remarks on Normal Forms

- Approach 1 is exponential in the worst case (e.g., transform $(a_1 \land b_1) \lor (a_2 \land b_2) \lor \cdots \lor (a_n \land b_n)$ to CNF).
- Approach 2 is polynomial
  - Basic idea: introduce labels for subformulas.
  - Also works for formulas with sharing (circuits).
  - Also known as “Tseitin Encoding”.
- CNF is usually not easier to solve, but easier to handle:
  - compact data structures: a CNF is simply a list of lists of literals.
- CNF very popular in practice: standard input format DIMACS
- To solve satisfiability of CNF, there are many optimization techniques and dedicated algorithms.