First Order Predicate Logic

Special Topics

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Special Topics

We will conclude by discussing the following special topics:

- the method of *induction* for reasoning about natural numbers,
- the expressiveness and limits of first-order predicate logic.

Mathematical Induction

A method to prove statements over the natural numbers.

- Goal: prove

$$\forall x \in \mathbb{N} : F$$

  i.e., formula F holds for all natural numbers.

- Rule:

$$\frac{K \ldots \vdash F[0/x] \quad K \ldots \vdash (\forall y \in \mathbb{N} : F[y/x] \rightarrow F[y+1/x])}{K \ldots \vdash \forall x \in \mathbb{N} : F}$$

$F[t/x]$: F where every free occurrence of x is replaced by t.

- Proof Steps:
  - *Induction base:* prove that F holds for 0.
  - *Induction hypothesis:* assume that F holds for new constant $\overline{x}$.
  - *Induction step:* prove that then F also holds for $\overline{x}+1$.

    Often the constant symbol x itself is chosen rather than $\overline{x}$.

Works because every natural number is reachable by a finite number of increments starting from 0.

Example

We prove the "sum of squares" formula

$$\forall n \in \mathbb{N} : \sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

by induction on n:

- Induction Base:

$$\sum_{i=1}^{0} i^2 = 0 = \frac{0 \cdot (0+1)(2 \cdot 0+1)}{6}$$

- Induction Hypothesis:

$$\sum_{i=1}^{\overline{n}} i^2 = \frac{\overline{n} \cdot (\overline{n}+1) \cdot (2\overline{n}+1)}{6} \qquad (*)$$

- Induction Step:

$$\begin{aligned}
\sum_{i=1}^{\overline{n}+1} i^2 &= (\overline{n}+1)^2 + \sum_{i=1}^{\overline{n}} i^2 \stackrel{(*)}{=} (\overline{n}+1)^2 + \frac{\overline{n} \cdot (\overline{n}+1) \cdot (2\overline{n}+1)}{6} \\
&= \frac{6(\overline{n}+1)^2 + \overline{n} \cdot (\overline{n}+1) \cdot (2\overline{n}+1)}{6} = \frac{(\overline{n}+1) \cdot (6 \cdot (\overline{n}+1) + \overline{n} \cdot (2\overline{n}+1))}{6} \\
&= \frac{(\overline{n}+1) \cdot (2\overline{n}^2 + 7\overline{n} + 6)}{6} = \frac{(\overline{n}+1) \cdot (\overline{n}+2) \cdot (2\overline{n}+3)}{6} \\
&= \frac{(\overline{n}+1) \cdot ((\overline{n}+1)+1) \cdot (2 \cdot (\overline{n}+1)+1)}{6} \qquad \square
\end{aligned}$$

Choice of Induction Variable

We define addition on $\mathbb{N}$ by primitive recursion:

$$x + 0 := x \tag{1}$$
$$x + (y + 1) := (x + y) + 1 \tag{2}$$

Our goal is to prove the associativity law

$$\forall x \in \mathbb{N}, y \in \mathbb{N}, z \in \mathbb{N} : x + (y + z) = (x + y) + z$$

For this purpose, we prove

$$\forall z \in \mathbb{N} : \underbrace{\forall x \in \mathbb{N}, y \in \mathbb{N} : x + (y + z) = (x + y) + z}_{F}$$

by induction on z.

Sometimes the appropriate choice of the induction variable is critical.

Choice of Induction Variable

We prove by induction on z

$$\forall z \in \mathbb{N} : \forall x \in \mathbb{N}, y \in \mathbb{N} : x + (y + z) = (x + y) + z$$

- Induction base: we prove

$$\forall x \in \mathbb{N}, y \in \mathbb{N} : x + (y + 0) = (x + y) + 0$$

We prove for arbitrary $x_0, y_0 \in \mathbb{N}$

$$x_0 + (y_0 + 0) \overset{(1)}{=} x_0 + y_0 \overset{(1)}{=} (x_0 + y_0) + 0$$

- Induction hypothesis (*): we assume

$$\forall x \in \mathbb{N}, y \in \mathbb{N} : x + (y + z) = (x + y) + z$$

- Induction step: we prove

$$\forall x \in \mathbb{N}, y \in \mathbb{N} : x + (y + (z + 1)) = (x + y) + (z + 1)$$

We prove for arbitrary $x_0, y_0 \in \mathbb{N}$

$$x_0 + (y_0 + (z + 1)) \overset{(2)}{=} x_0 + ((y_0 + z) + 1) \overset{(2)}{=} (x_0 + (y_0 + z)) + 1$$
$$\overset{(*)}{=} ((x_0 + y_0) + z) + 1 \overset{(2)}{=} (x_0 + y_0) + (z + 1) \quad \square$$

Induction with a Different Starting Value

- Goal: prove
$$\forall x \in \mathbb{N} : x \geq b \rightarrow F$$
i.e., formula F holds for all natural numbers greater than or equal to some natural number b.
- Rule:
$$\frac{K \ldots \vdash F[b/x] \quad K \ldots \vdash (\forall y \in \mathbb{N} : y \geq b \wedge F[y/x] \rightarrow F[y+1/x])}{K \ldots \vdash (\forall x \in \mathbb{N} : x \geq b \rightarrow F)}$$
- Proof Steps:
  - *Induction base:* prove that F holds for b.
  - *Induction hypothesis:* assume that F holds for $\overline{x} \geq b$.
  - *Induction step:* prove that then F also holds for $\overline{x}+1$.

Induction works with arbitrary starting values.

Example

We prove

$$\forall n \in \mathbb{N} : n \geq 4 \rightarrow n^2 \leq 2^n$$

- Induction base: we show

$$4^2 = 16 = 2^4$$

- Induction hypothesis: we assume for $n \geq 4$

$$n^2 \leq 2^n \qquad (*)$$

- Induction step: we show

$$(n+1)^2 = n^2 + 2n + 1 \overset{1 \leq n}{\leq} n^2 + 2n + n = n^2 + 3n \overset{0 \leq n}{\leq} n^2 + 4n$$
$$\overset{4 \leq n}{\leq} n^2 + n \cdot n = n^2 + n^2 = 2n^2 \overset{(*)}{\leq} 2 \cdot 2^n = 2^{n+1} \quad \square$$

Complete Induction

A generalized form of the induction method.

- Rule:

$$\frac{K \ldots \vdash (\forall x \in \mathbb{N} : (\forall y \in \mathbb{N} : y < x \rightarrow F[y/x]) \rightarrow F)}{K \ldots \vdash \forall x \in \mathbb{N} : F}$$

- Proof steps:
  - *Induction hypothesis:* assume that F holds for all y less than $\overline{x}$.
  - *Induction step:* prove that F then also holds for $\overline{x}$.

The induction assumption is applied not only to the direct predecessor.

Example

We take function $T : \mathbb{N} \to \mathbb{N}$ where

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2 \cdot T(n/2) & \text{if } n > 0 \wedge 2|n \\ 1 + 2 \cdot T((n-1)/2) & \text{else} \end{cases}$$

and prove by complete induction on n

$$\forall n \in \mathbb{N} : T(n) = n$$

- Induction hypothesis:

$$\forall m \in \mathbb{N} : m < n \to T(m) = m \qquad (*)$$

- Induction step:
  - Case $n = 0$: we know $T(n) = T(0) = 0 = n$
  - Case $n > 0 \wedge 2|n$: we know

$$T(n) = 2 \cdot T(n/2) \overset{(*)}{=} 2 \cdot (n/2) = n$$

  - Case $n > 0 \wedge \neg(2|n)$: we know

$$T(n) = 1 + 2 \cdot T((n-1)/2) \overset{(*)}{=} 1 + 2 \cdot ((n-1)/2) = 1 + (n-1) = n \quad \square$$

Expressiveness of First-Order Logic

- Variables denote elements of the domain, thus no quantification is possible over functions and predicates of the domain.

  This would require second-order predicate logic.

- Nevertheless we express in first-order logic statements such as

  $$\forall A, B, f \in A \to B : f \text{ is bijective} \to \exists g \in B \to A : \forall x \in B : f(g(x)) = x$$

- This is possible because formulas are usually interpreted over the domain of sets, i.e., all variables denote sets:

  $$\begin{aligned} A \to B := \{S \subseteq A \times B \mid \\ & (\forall a \in A : \exists b \in B : (a,b) \in S) \wedge \\ & (\forall a, a', b : (a,b) \in S \wedge (a',b) \in S \to a = a')\} \end{aligned}$$

- Terms like $f(g(x))$ involve a hidden binary function "apply"

  $$f(g(x)) \rightsquigarrow apply(f, apply(g, x))$$

  which denotes "function application":

  $$apply(f, x) := \textbf{the } y : (x, y) \in f$$

First-order predicate logic over the domain of sets is the "working horse" of mathematics; virtually all of mathematics is formulated in this framework.

Soundness and Completeness of First-Order Logic

Now we turn our attention to the second question.

Completeness Theorem (Kurt Gödel, 1929): First order predicate logic has a proof calculus for which the following holds:

- Soundness: if by the rules of the calculus a conclusion F can be derived from a set of assumptions Γ ($\Gamma \vdash F$), then F is a logical consequence of Γ ($\Gamma \models F$).
- Completeness: if F is a logical consequence of Γ ($\Gamma \models F$), then by the rules of the calculus F can be derived from Γ ($\Gamma \vdash F$).

No logic that is stronger (more expressive) than first order predicate logic has a proof calculus that also enjoys both soundness and completeness.

Undecidability of First-Order Logic

The existence of a complete proof calculus does not mean that the truth of every formula is algorithmically decidable.

- Undecidability (Church/Turing, 1936/1937): there does not exist any algorithm that for given formula set Γ and formula F always terminates and says whether $\Gamma \models F$ holds or not.
- Semidecidability: but there exists an algorithm, that for given Γ and F, if $\Gamma \models F$, detects this fact in a finite amount of time.

  This algorithm searches for a proof of $\Gamma \vdash F$ in a complete proof calculus; if such a proof exists, it will eventually detect it; however, if no such proof exists, the search runs forever.

Automatic proof search is not able to detect that a formula is not true.

Limits of First-Order Logic

Not every structure can be completely described by a finite set of formulas.

- Incompleteness Theorem (Kurt Gödel, 1931): it is in no sound logic possible to prove all true arithmetic statements (i.e., all statements about natural numbers with addition and multiplication).
  - To adequately characterize $\mathbb{N}$, the (infinite) axiom scheme of mathematical induction has to be added.
- Corollary: in every sound formal system that is sufficiently rich there are statements that can neither be proved nor disproved.

In practice, complete reasoners for first-order logic are often supported by (complete or incomplete) reasoners for special theories.