

LOGIC

PROPOSITIONAL REASONING

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Satisfiability Checking

Definition (Satisfiability Problem of Propositional Logic (SAT))

Given a formula ϕ , is there an assignment ν such that $[\phi]_{\nu} = \mathbf{1}$?

- oldest **NP**-complete problem (see next slides)
 - checking a solution (assignment satisfies formula) is easy (polynomial effort)
 - finding a solution is difficult (probably exponential in the worst case)
- many practical applications (used in industry)
- efficient SAT solvers (solving tools) are available
- other problems can be translated to SAT:

problem	formulation in propositional logic
ϕ is valid	$\neg\phi$ is unsatisfiable
ϕ is refutable	to $\neg\phi$ is satisfiable
$\phi \Leftrightarrow \psi$	to $\neg(\phi \leftrightarrow \psi)$ is unsatisfiable
$\phi_1, \dots, \phi_n \models \psi$	$\phi_1 \wedge \dots \wedge \phi_n \wedge \neg\psi$ is unsatisfiable

A Glimpse of Complexity Theory

- characterization of computational **hardness** of a problem
- **Turing Machine**: machine model for abstract “run time” or “memory usage”
- the focus is on worst-case **asymptotic** time and space usage

Definition

problem is in $\mathcal{O}(f(n))$ iff exists constant c and an algorithm which needs $c \cdot f(n)$ steps (in the worst case on a Turing machine) for an input of size n

- logarithmic $\mathcal{O}(\log n)$, e.g. binary search on sorted array of size n
- linear $\mathcal{O}(n)$, e.g. linear search in list with n elements
- quadratic $\mathcal{O}(n^2)$, e.g. generate list of pairs of n elements
- exponential $\mathcal{O}(2^n)$, e.g. produce all subsets of a set of n elements

Definition

polynomial problems: exists k such that worst-case run time is in $\mathcal{O}(n^k)$
class of polynomial problems is called **P**

SAT and the Complexity Class NP

Definition

A decision problem asks whether an input belongs to a certain class.

Prime: decide whether a number given as input is prime.

SAT: decide whether formula given as input is satisfiable.

Basic idea of **NP** is to use a “guess” and “check” approach, where “guessing” is non-deterministic, e.g. just a “good” choice has to exist.

Definition

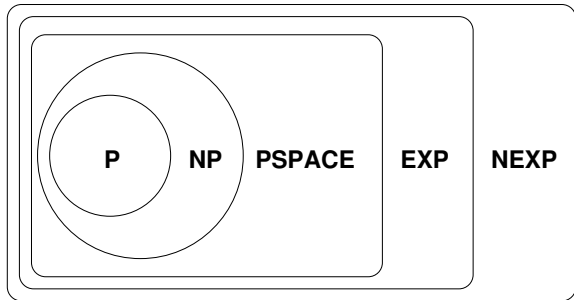
The class NP contains all decision problems which can be decided by a “guessing” and “checking” algorithm in polynomial time in the input size.

Clearly both **Prime** and **SAT** belong to **NP**.

Theorem (Cook'71)

*Any decision problem in **NP** can be reduced (encoded) polynomially into **SAT**.*

Complexity Hierarchy



P polynomial time

NP non-deterministic polynomial time

PSPACE polynomial space

EXP exponential time

NEXP non-deterministic exponential time

except for $P \neq EXP$ and $NP \neq NEXP$ nothing is known about strict inclusion

Simple Algorithm for Satisfiability Checking

1 **Algorithm:** evaluate

Data: formula ϕ

Result: 1 iff ϕ is satisfiable

2 **if** ϕ contains a variable x **then**

3 | pick $v \in \{\top, \perp\}$

4 | /* replace x by truth constant v , evaluate resulting formula */

5 | **if** $evaluate(\phi[x|v])$ **then return 1;**

6 | **else return** $evaluate(\phi[x|\bar{v}])$;

7 **else**

8 | **switch** ϕ **do**

9 | **case** \top **do return 1;**

10 | **case** \perp **do return 0;**

11 | **case** $\neg\psi$ **do return** $! evaluate(\psi)$ /* true iff ψ is false */ ;

12 | **case** $\psi' \wedge \psi''$ **do**

13 | | **return** $evaluate(\psi') \ \&\& \ evaluate(\psi'')$ /* true iff both ψ' and ψ'' are true */

14 | **case** $\psi' \vee \psi''$ **do**

15 | | **return** $evaluate(\psi') \ || \ evaluate(\psi'')$ /* true iff ψ' or ψ'' is true */

Reasoning with (Propositional) Calculi

- **goal:** automatically reason about (propositional) formulas
i.e., mechanically show validity / unsatisfiability
- **basic idea:** use syntactical manipulations to prove/refute a formula
- **elements of a calculus:**
 - axioms:** trivial truths/trivial contradictions
 - rules:** inference of new formulas
- **approach:** construct a **proof/refutation**
 - apply the rules of the calculus until only axioms are inferred
 - if this is not possible, then the formula is not valid/unsatisfiable
- **examples of calculi:**
 - sequence calculus: shows validity (actually entailment)
 - resolution calculus: shows unsatisfiability

Logic Entailment

copied from previous slide set

Let $\phi_1, \dots, \phi_n, \psi$ be propositional formulas.

Then ϕ_1, \dots, ϕ_n **entail** ψ (written as $\phi_1, \dots, \phi_n \models \psi$) iff

$[\phi_1]_\nu = \mathbf{1}, \dots, [\phi_n]_\nu = \mathbf{1}$ implies that $[\psi]_\nu = \mathbf{1}$.

Informal meaning: True premises derive a true conclusion.

- \models is a *meta-symbol* (it is not part of the language)
- $\phi_1, \dots, \phi_n \models \psi$ iff $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \psi$ is valid, i.e., we can express semantics by means of syntactics.
- If ϕ_1, \dots, ϕ_n do not entail ψ , we write $\phi_1, \dots, \phi_n \not\models \psi$.

Example:

■ $a \models a \vee b$

■ $\models a \vee \neg a$

■ $a, a \rightarrow b \models b$

■ $a, b \models a \wedge b$

■ $\not\models a \wedge \neg a$

■ $\perp \models a \wedge \neg a$

Sequents

Definition

A **sequent** is an expression of the form

$$\phi_1, \dots, \phi_n \vdash \psi$$

where $\phi_1, \dots, \phi_n, \psi$ are propositional formulas.

The formulas ϕ_1, \dots, ϕ_n are called **assumptions**, ψ is called **goal**.

remarks:

- **intuitively** $\phi_1, \dots, \phi_n \vdash \psi$ means goal ψ follows from $\{\phi_1, \dots, \phi_n\}$
- **special case** $n = 0$:
 - written as $\vdash \psi$
 - meaning: we have to prove that ψ is valid
- **notation**: for sequent $\phi_1, \dots, \phi_n \vdash \psi$, we write $K \dots \phi_i \vdash \psi$ if we are only interested in assumption ϕ_i
- the assumptions are **orderless** not ordered

Axiom and Structural Rules

- axiom **"goal in assumption"**:

If the goal is among the assumptions, the goal can be proved.

$$\text{GoalAssum} \frac{}{K \dots, \psi \vdash \psi}$$

- axiom **"contradiction in assumptions"**:

If the assumptions are contradicting, anything can be proved.

$$\text{ContrAssum} \frac{}{K \dots, \phi, \neg\phi \vdash \psi}$$

- rule **"add valid assumption"**:

$$\text{ValidAssum} \frac{K \dots, \phi \vdash \psi}{K \dots \vdash \psi} \text{ if } \phi \text{ is valid}$$

Negation Rules

■ rules "contradiction":

$$A_{\neg} \frac{K \dots \neg \psi \vdash \perp}{K \dots \vdash \psi}$$

$$P_{\neg} \frac{K \dots \vdash \neg \phi}{K \dots, \phi \vdash \perp}$$

■ rules "elimination of double negation":

$$P_{\neg_d} \frac{K \dots \vdash \psi}{K \dots \vdash \neg \neg \psi}$$

$$A_{\neg_d} \frac{K \dots, \phi \vdash \psi}{K \dots, \neg \neg \phi \vdash \psi}$$

Binary Connective Rules

■ rules "conjunction":

$$A-\wedge \frac{K \dots, \phi_1, \phi_2 \vdash \psi}{K \dots, \phi_1 \wedge \phi_2 \vdash \psi}$$

$$P-\wedge \frac{K \dots \vdash \psi_1 \quad K \dots \vdash \psi_2}{K \dots \vdash \psi_1 \wedge \psi_2}$$

■ rules "disjunction":

$$P-\vee \frac{K \dots, \neg \psi_1 \vdash \psi_2}{K \dots \vdash \psi_1 \vee \psi_2}$$

$$A-\vee \frac{K \dots, \phi_1 \vdash \psi \quad K \dots, \phi_2 \vdash \psi}{K \dots, \phi_1 \vee \phi_2 \vdash \psi}$$

Rules for other connectives like implication " \rightarrow " and equivalence " \leftrightarrow " are constructed accordingly.

Some Remarks on Sequent Calculus

- **premises** of a rule: sequent(s) above the line
- **conclusion** of a rule: sequent below the line
- **axiom**: rule without premises
- **non-deterministic rule**: $P\text{-}\vee$
- **further non-determinism**: decision which rule to apply next
- **rules with case split**: $P\text{-}\wedge$, $A\text{-}\vee$
- **proof of formula ψ**
 1. start with $\vdash \psi$
 2. apply rules from bottom to top as long as possible, i.e., for given conclusion, find suitable premise(s)
 3. if finally all sequents are axioms then ψ is valid
- **note**: there are many variants of the sequent calculus

Computing with Sequent Calculus

1 Algorithm: entails

Data: set of assumptions \mathcal{A} , formula ψ

Result: 1 iff \mathcal{A} entails ψ , i.e., $\mathcal{A} \models \psi$

```
2 if  $\psi = \neg\neg\psi'$  then return entails ( $\mathcal{A}, \psi'$ );
3 if  $\neg\neg\phi \in \mathcal{A}$  then return entails ( $\mathcal{A} \setminus \{\neg\neg\phi\} \cup \{\phi\}, \psi$ );
4 if  $\phi_1 \wedge \phi_2 \in \mathcal{A}$  then return entails ( $\mathcal{A} \setminus \{\phi_1 \wedge \phi_2\} \cup \{\phi_1, \phi_2\}, \psi$ );
5 if  $(\psi \in \mathcal{A})$  or  $(\phi, \neg\phi \in \mathcal{A})$  then return 1;
6 if  $\mathcal{A} \cup \{\psi\}$  contains only literals then return 0;
7 switch  $\psi$  do
8   case  $\perp$  do
9     if  $\neg\phi \in \mathcal{A}$  then return entails ( $\mathcal{A} \setminus \{\neg\phi\}, \phi$ );
10    if  $\phi_1 \vee \phi_2 \in \mathcal{A}$  then
11      if ! entails ( $\mathcal{A} \setminus \{\neg\phi_1 \vee \phi_2\} \cup \{\phi_1\}, \perp$ ) then return 0;
12      else return entails ( $\mathcal{A} \setminus \{\neg\phi_1 \vee \phi_2\} \cup \{\phi_2\}, \perp$ );
13   case  $x$  where  $x$  is a variable do return entails ( $\mathcal{A} \cup \{\neg x\}, \perp$ );
14   case  $\neg\psi'$  do return entails ( $\mathcal{A} \cup \{\psi'\}, \perp$ );
15   case  $\psi_1 \vee \psi_2$  do return entails ( $\mathcal{A} \cup \{\neg\psi_1\}, \psi_2$ );
16   case  $\psi_1 \wedge \psi_2$  do return entails ( $\mathcal{A}, \psi_1$ ) && entails ( $\mathcal{A}, \psi_2$ );
```

Proving XOR stronger than OR



$$\begin{array}{c}
 \text{GoalAssum} \frac{}{b, (\neg a \vee \neg b), \neg a \vdash b} \quad \text{ContrAssum} \frac{}{a, (\neg a \vee \neg b), \neg a \vdash b} \\
 \text{A-}\vee \frac{}{\frac{}{(a \vee b), (\neg a \vee \neg b), \neg a \vdash b}}{(a \vee b), (\neg a \vee \neg b) \vdash a \vee b}} \\
 \text{P-}\vee \frac{}{\frac{}{(a \vee b) \wedge (\neg a \vee \neg b) \vdash a \vee b}}{\neg \neg ((a \vee b) \wedge (\neg a \vee \neg b)) \vdash a \vee b}} \\
 \text{A-}\neg a \frac{}{} \\
 \text{P-}\vee \frac{}{\vdash \neg ((a \vee b) \wedge (\neg a \vee \neg b)) \vee (a \vee b)}
 \end{array}$$

proof direction

Refuting XOR stronger than AND

$$\begin{array}{c}
 \text{GAss} \frac{}{a, (\neg a \vee \neg b) \vdash a} \quad \text{A-}\vee \frac{\text{CAss} \frac{}{b, \neg b \vdash a} \quad b, \neg a \vdash a}{b, (\neg a \vee \neg b) \vdash a}}{(a \vee b), (\neg a \vee \neg b) \vdash a} \quad \text{A-}\vee \frac{\vdots \quad \vdots}{(a \vee b), (\neg a \vee \neg b) \vdash b} \\
 \text{P-}\wedge \frac{}{(a \vee b), (\neg a \vee \neg b) \vdash a} \quad \text{A-}\wedge \frac{(a \vee b), (\neg a \vee \neg b) \vdash a \wedge b}{(a \vee b) \wedge (\neg a \vee \neg b) \vdash a \wedge b} \\
 \text{A-}\neg_d \frac{}{\neg\neg((a \vee b) \wedge (\neg a \vee \neg b)) \vdash a \wedge b} \\
 \text{P-}\vee \frac{}{\vdash \neg((a \vee b) \wedge (\neg a \vee \neg b)) \vee (a \wedge b)}
 \end{array}$$

counter example to validity: $a = \perp, b = \top$

Soundness and Completeness

For any calculus important properties are,

soundness, i.e. the question “**Can only valid formulas be shown as valid?**” and **completeness**, i.e. the question “**Is there a proof for every valid formula?**”.

Soundness

If a formula is shown to be valid in the Gentzen Calculus, then it is valid.

Proof sketch:

Consider each rule individually and show that from valid premises only valid conclusions can be drawn.

Completeness

Every valid formula can be proven to be valid in the Gentzen Calculus.

Proof sketch:

Show algorithm terminates and that there is at least one case where it returns false if the formula is not valid.

Proving Formulas in Normal Form

- In practice, formulas of arbitrary structure are quite challenging to handle
 - tree structure
 - simplifications affect only subtrees
- We have seen that CNF and DNF are able to represent every formula
 - so why not use them as input for SAT?
- **Conjunctive Normal Form**
 - refutability is easy to show
 - CNF can be efficiently calculated (polynomial)
- **Disjunctive Normal Form**
 - satisfiability is easy to show
 - complexity is in getting the DNF
- CNF and DNF can be obtained from the **truth tables**
 - exponential many assignments have to be considered
- alternative approach
 - structural rewritings** are (satisfiability) equivalence preserving

Transformation to Normal Form

1. Remove \leftrightarrow , \rightarrow , \oplus as follows:

$$\phi \leftrightarrow \psi \Leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi), \phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi,$$

$$\phi \oplus \psi \Leftrightarrow (\phi \vee \psi) \wedge (\neg\phi \vee \neg\psi)$$

2. Transform formula to negation normal form (NNF) by application of laws of De Morgan and elimination of double negation
3. Transform formula to CNF (DNF) by laws of distributivity

Example

Transform $\neg(a \leftrightarrow b) \rightarrow (\neg(c \wedge d) \wedge e)$ to an equivalent formula in CNF.

1. a) remove equivalences: $\Leftrightarrow \neg((a \rightarrow b) \wedge (b \rightarrow a)) \rightarrow (\neg(c \wedge d) \wedge e)$
b) remove implications: $\Leftrightarrow \neg\neg((\neg a \vee b) \wedge (\neg b \vee a)) \vee (\neg(c \wedge d) \wedge e)$
2. NNF: $\Leftrightarrow ((\neg a \vee b) \wedge (\neg b \vee a)) \vee ((\neg c \vee \neg d) \wedge e)$
3. $\Leftrightarrow ((\neg a \vee b) \vee ((\neg c \vee \neg d) \wedge e)) \wedge ((\neg b \vee a) \vee ((\neg c \vee \neg d) \wedge e))$
 $\Leftrightarrow (\neg a \vee b \vee \neg c \vee \neg d) \wedge (\neg a \vee b \vee e) \wedge (\neg b \vee a \vee \neg c \vee \neg d) \wedge (\neg b \vee a \vee e)$

Some Remarks on Normal Forms

- The presented transformation to CNF/DNF is exponential in the worst case (e.g., transform $(a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \dots \vee (a_n \wedge b_n)$ to CNF).
- For DNF transformation, there is probably no better algorithm.
- For CNF transformation, there are polynomial algorithms.
 - Basic idea: introduce labels for subformulas.
 - Also works for formulas with sharing (circuits).
 - Also known as “Tseitin Encoding”.
- CNF is usually not easier to solve, but easier to handle:
 - compact data structures: a CNF is simply a list of lists of literals.
- CNF very popular in practice: standard input format DIMACS
- To solve satisfiability of CNF, there are many optimization techniques and dedicated algorithms.

Resolution

- the **resolution calculus** consists of the single resolution rule

$$\frac{x \vee C \quad \neg x \vee D}{C \vee D}$$

- C and D are (possibly empty) clauses
- the clause $C \vee D$ is called **resolvent**
- variable x is called **pivot**
- usually antecedent clauses $x \vee C$ and $\neg x \vee D$ are assumed not to be tautological, i.e., $x \notin C$ and $x \notin D$.

- in other words:

$$(\neg x \rightarrow C), (x \rightarrow D) \models C \vee D$$

- resolution is **sound** and **complete**.

- the resolution calculus works only on formulas in CNF
- if the empty clause can be derived then the formula is **unsatisfiable**
- if no new clause can be generated by application of the resolution rule then the formula is **satisfiable**

Example

one application of resolution

$$\frac{x \vee y \vee \neg z \quad \neg x \vee y' \vee \neg z}{y \vee \neg z \vee y'}$$

derivation of empty clause:

$$\frac{y \quad \neg y}{\perp}$$

derivation of tautology:

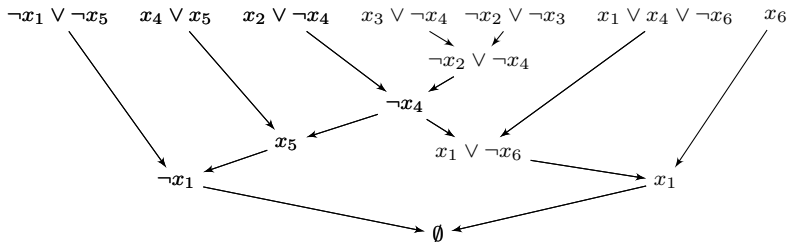
$$\frac{x \vee a \quad \neg x \vee \neg a}{a \vee \neg a}$$

Resolution Example

We prove unsatisfiability of

$$\{(\neg x_1 \vee \neg x_5), (x_4 \vee x_5), (x_2 \vee \neg x_4), (x_3 \vee \neg x_4), (\neg x_2 \vee \neg x_3), (x_1 \vee x_4 \vee \neg x_6), (x_6)\}$$

as follows:



DPLL Overview

The DPLL algorithm is ...

- **old** (invented 1962)
- **easy** (basic pseudo-code is less than 10 lines)
- **popular** (well investigated; also theoretical properties)
- usually realized for **formulas in CNF**
- using **binary constraint propagation (BCP)**
- in its modern form as **conflict drive clause learning (CDCL)**
basis for state-of-the-art SAT solvers

Binary Constraint Propagation

Definition (Binary Constraint Propagation (BCP))

Let ϕ be a formula in CNF containing a unit clause C , i.e., ϕ has a clause $C = (l)$ which consists only of literal l . Then $BCP(\phi, l)$ is obtained from ϕ by

- removing all clauses with l
- removing all occurrences of \bar{l}

- BCP on variable x can trigger application of BCP on variable y
- if BCP produces the empty clause, then the formula is unsatisfiable
- if BCP produces the empty CNF, then the formula is satisfiable

Example

$\phi = \{(\neg a \vee b \vee \neg c), (a \vee b), (\neg a \vee \neg b), (a)\}$

1. $\phi' = BCP(\phi, a) = \{(b \vee \neg c), (\neg b)\}$

2. $\phi'' = BCP(\phi', \neg b) = \{(\neg c)\}$

3. $\phi''' = BCP(\phi'', c) = \{\} = \top$

DPLL Algorithm

1 **Algorithm:** evaluate

Data: formula ϕ in CNF

Result: 1 iff ϕ satisfiable

2 **while** 1 **do**

```
3   |  $\phi = \text{BCP}(\phi)$ 
4   | if  $\phi == \top$  then return 1;
5   | if  $\phi == \perp$  then
6   |   | if stack.isEmpty() then return 0;
7   |   |  $(l, \phi) = \text{stack.pop}()$ 
8   |   |  $\phi = \phi \wedge l$ 
9   | else
10  |   | select literal  $l$  occurring in  $\phi$ 
11  |   |  $\text{stack.push}(\bar{l}, \phi)$ 
12  |   |  $\phi = \phi \wedge l$ 
```

Some Remarks on DPLL

- DPLL is the basis for most state-of-the-art SAT solvers

- Lingeling <http://fmv.jku.at/lingeling>
- CaDiCaL <http://fmv.jku.at/cadical>
- some more established solvers: MiniSAT, PicoSAT, Glucose, . . .

- DPLL alone is not enough - powerful optimizations required for efficiency:

- learning and non-chronological back-tracking (CDCL)
- reset strategies and phase-saving
- compact lazy data-structures
- variable selection heuristics
- usually combined with preprocessing before and inprocessing during search

- variants of DPLL are also used for other logics:

- quantified propositional logic (QBF)
- satisfiability modulo theories (SMT)

- challenge to parallelize

- some successful attempts: ManySAT, Plingeling, Penelope, Treengeling, . . .