PROPOSITIONAL LOGIC

VL Logik: WS 2019/20

(Version 2019.1)

Martina Seidl (martina.seidl@jku.at),
Armin Biere (biere@jku.at)
Institut für Formale Modelle und Verifikation

JOHANNES KEPLER
UNIVERSITY LINZ

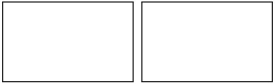
The Box Game

Game Board:

The Box Game

Game Board:

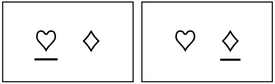
- The game board consists of **boxes**.



The Box Game

Game Board:

- The game board consists of boxes.
- The boxes contain **symbols**.



The Box Game

Game Board:

- The game board consists of boxes.
- The boxes contain symbols.
- Some symbols are **underlined**.

$$\boxed{\underline{\heartsuit}\quad\diamondsuit}\ \boxed{\heartsuit\quad\underline{\diamondsuit}}$$

The Box Game: Rules

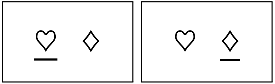
Rules of the Game:

$$\boxed{\;\underline{\heartsuit}\quad \diamondsuit\;}\ \boxed{\;\heartsuit\quad \underline{\diamondsuit}\;}$$

The Box Game: Rules

Rules of the Game:

- There are two colors, for example blue and red.
  - One color is the winning color, for example red.
  - Then the non-winning color is blue.



The Box Game: Rules

Rules of the Game:

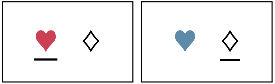
- There are two colors, for example blue and red.
  - One color is the winning color, for example red.
  - Then the non-winning color is blue.
- Now we can play: assign the two colors to each symbol such that its underlined and non-underlined occurrences have a different color.

$$\boxed{\underline{\heartsuit}\quad\diamondsuit}\ \boxed{\heartsuit\quad\underline{\diamondsuit}}$$

The Box Game: Rules

Rules of the Game:

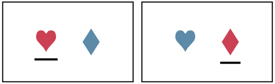
- There are two colors, for example blue and red.
  - One color is the winning color, for example red.
  - Then the non-winning color is blue.
- Now we can play: assign the two colors to each symbol such that its underlined and non-underlined occurrences have a different color.



The Box Game: Rules

Rules of the Game:

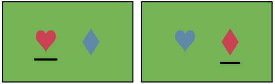
- There are two colors, for example blue and red.
  - One color is the winning color, for example red.
  - Then the non-winning color is blue.
- Now we can play: assign the two colors to each symbol such that its underlined and non-underlined occurrences have a different color.



The Box Game: Rules

Rules of the Game:

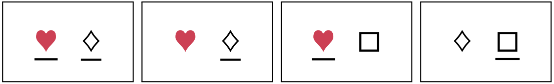
- There are two colors, for example blue and red.
  - One color is the winning color, for example red.
  - Then the non-winning color is blue.
- Now we can play: assign the two colors to each symbol such that its underlined and non-underlined occurrences have a different color.
- If each box contains at least one symbol in the winning color, you won.



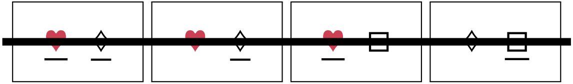
Some Examples

$\boxed{\underline{\heartsuit} \quad \underline{\diamondsuit}}$ $\boxed{\heartsuit \quad \underline{\diamondsuit}}$ $\boxed{\underline{\heartsuit} \quad \square}$ $\boxed{\diamondsuit \quad \underline{\square}}$

Some Examples

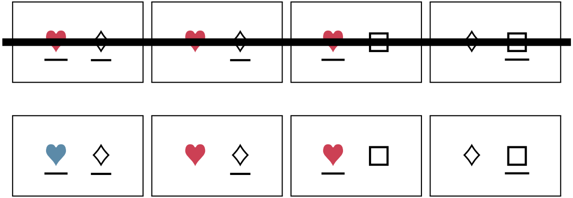


Some Examples



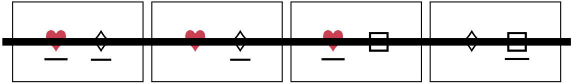
Wrong!

Some Examples

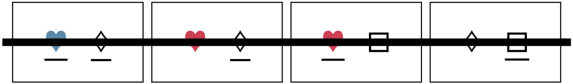


Wrong!

Some Examples

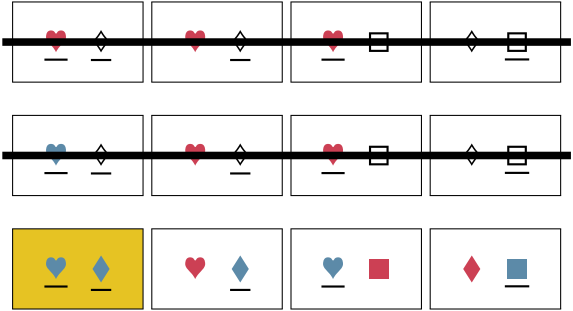


Wrong!



Wrong!

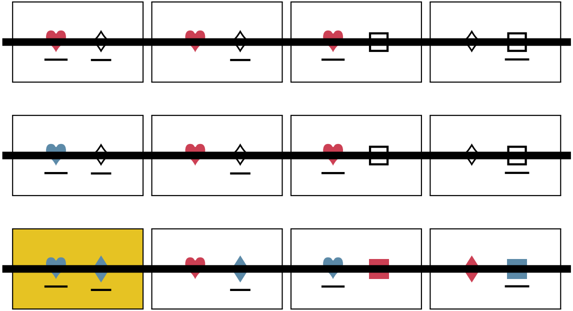
Some Examples



Wrong!

Wrong!

Some Examples

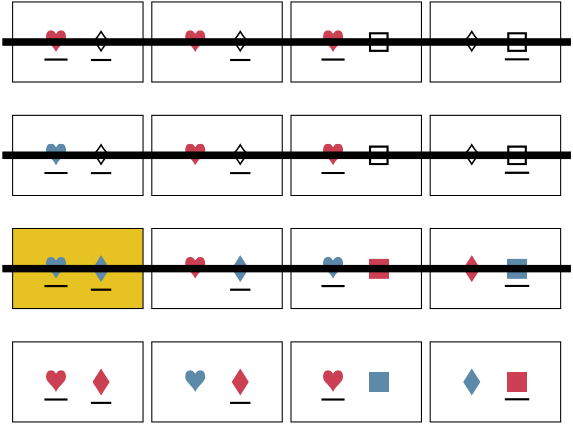


Wrong!

Wrong!

Lost!

Some Examples

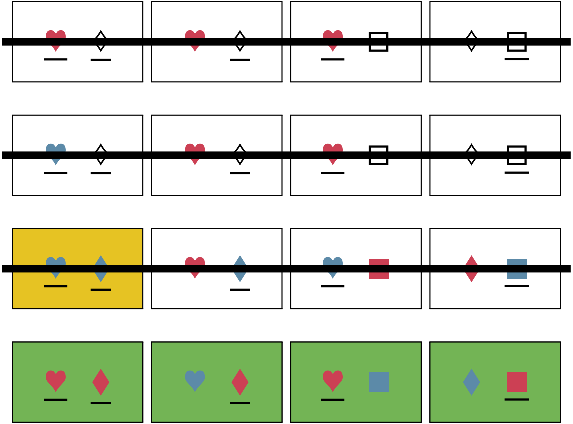


Wrong!

Wrong!

Lost!

Some Examples



Wrong!

Wrong!

Lo

Some Terminology

- From now on, we call a box a **clause**.
- We call a clause with at least one red symbol **satisfied**.
- We call a clause with all symbols in blue **falsified**.
- We call a clause with blue and uncolored symbols **undecided**.

⇒ The game is won if all clauses are satisfied.

How Many Possibilities?

- 1 symbol, 2 possibilities
  1. ♥
  2. ♥

How Many Possibilities?

- 1 symbol, 2 possibilities
  1. ♥
  2. ♥
- 2 symbols, 4 possibilities
  1. ♥, ♦
  2. ♥, ♦
  3. ♥, ♦
  4. ♥, ♦

How Many Possibilities?

- 1 symbol, 2 possibilities
  1. $\textcolor{red}{\heartsuit}$
  2. $\textcolor{blue}{\heartsuit}$
- 2 symbols, 4 possibilities
  1. $\textcolor{red}{\heartsuit}$, $\textcolor{red}{\diamondsuit}$
  2. $\textcolor{red}{\heartsuit}$, $\textcolor{blue}{\diamondsuit}$
  3. $\textcolor{blue}{\heartsuit}$, $\textcolor{red}{\diamondsuit}$
  4. $\textcolor{blue}{\heartsuit}$, $\textcolor{blue}{\diamondsuit}$
- 3 symbols, 8 possibilities
  1. $\textcolor{red}{\heartsuit}$, $\textcolor{red}{\diamondsuit}$, $\textcolor{red}{\blacksquare}$
  2. $\textcolor{red}{\heartsuit}$, $\textcolor{blue}{\diamondsuit}$, $\textcolor{red}{\blacksquare}$
  3. $\textcolor{blue}{\heartsuit}$, $\textcolor{red}{\diamondsuit}$, $\textcolor{red}{\blacksquare}$
  4. $\textcolor{blue}{\heartsuit}$, $\textcolor{blue}{\diamondsuit}$, $\textcolor{red}{\blacksquare}$
  5. $\textcolor{red}{\heartsuit}$, $\textcolor{red}{\diamondsuit}$, $\textcolor{blue}{\blacksquare}$
  6. $\textcolor{red}{\heartsuit}$, $\textcolor{blue}{\diamondsuit}$, $\textcolor{blue}{\blacksquare}$
  7. $\textcolor{blue}{\heartsuit}$, $\textcolor{red}{\diamondsuit}$, $\textcolor{blue}{\blacksquare}$
  8. $\textcolor{blue}{\heartsuit}$, $\textcolor{blue}{\diamondsuit}$, $\textcolor{blue}{\blacksquare}$

How Many Possibilities?

- 1 symbol, 2 possibilities
    1. ♥
    2. ♥
- 2 symbols, 4 possibilities
    1. ♥, ♦
    2. ♥, ♦
    3. ♥, ♦
    4. ♥, ♦
- 3 symbols, 8 possibilities
    1. ♥, ♦, ■
    2. ♥, ♦, ■
    3. ♥, ♦, ■
    4. ♥, ♦, ■
    5. ♥, ♦, ■
    6. ♥, ♦, ■
    7. ♥, ♦, ■
    8. ♥, ♦, ■
- 4 symbols, 16 possibilities

How Many Possibilities?

- 1 symbol, 2 possibilities
  1. ♥
  2. ♥
- 2 symbols, 4 possibilities
  1. ♥, ♦
  2. ♥, ♦
  3. ♥, ♦
  4. ♥, ♦
- 3 symbols, 8 possibilities
  1. ♥, ♦, ■
  2. ♥, ♦, ■
  3. ♥, ♦, ■
  4. ♥, ♦, ■
  5. ♥, ♦, ■
  6. ♥, ♦, ■
  7. ♥, ♦, ■
  8. ♥, ♦, ■
- 4 symbols, 16 possibilities
- ...

How Many Possibilities?

- 1 symbol, 2 possibilities
  1. ♥
  2. ♥
- 2 symbols, 4 possibilities
  1. ♥, ♦
  2. ♥, ♦
  3. ♥, ♦
  4. ♥, ♦
- 3 symbols, 8 possibilities
  1. ♥, ♦, ■
  2. ♥, ♦, ■
  3. ♥, ♦, ■
  4. ♥, ♦, ■
  5. ♥, ♦, ■
  6. ♥, ♦, ■
  7. ♥, ♦, ■
  8. ♥, ♦, ■
- 4 symbols, 16 possibilities
- ...

n symbols

$\Rightarrow$

2^n possibilities

Guess & Check Problems

observation in our BOX game:

- finding a solution is hard
  - 2^n solution candidates have to be considered
  - a good oracle is needed for guessing
- verifying a given candidate solution is easy
  - check that each box contains a red symbol

Guess & Check Problems

observation in our BOX game:

- finding a solution is hard
  - 2^n solution candidates have to be considered
  - a good oracle is needed for guessing
- verifying a given candidate solution is easy
  - check that each box contains a red symbol

fundamental question in computer science:

The P = NP Question

Is searching for a solution harder than verifying a solution?

(unfortunately, the answer is not known)

Famous Guess & Check Problem: SAT

SAT is the decision problem of propositional logic:

- Given a Boolean formula, for example

$$(\neg x \vee \neg y) \wedge (x \vee \neg y) \wedge (\neg x \vee z) \wedge (y \vee \neg z).$$

- Question: is the formula satisfiable?
  I.e., is there an assignment of truth values **1** (true), **0** (false) to the literals $x, y, z, \neg x, \neg y, \neg z$ such that
  - for every variable $v \in \{x, y, z\}$ it holds that the truth value of v and the truth value of $\neg v$ are different
  - each clause $(...)$ contains at least one true literal

Famous Guess & Check Problem: SAT

SAT is the decision problem of propositional logic:

- Given a Boolean formula, for example

$$(\neg x \lor \neg y) \land (x \lor \neg y) \land (\neg x \lor z) \land (y \lor \neg z).$$

- Question: is the formula satisfiable?
  I.e., is there an assignment of truth values **1** (true), **0** (false) to the literals $x, y, z, \neg x, \neg y, \neg z$ such that
  - for every variable $v \in \{x, y, z\}$ it holds that the truth value of v and the truth value of $\neg v$ are different
  - each clause $(\ldots)$ contains at least one true literal

Cook-Levin Theorem [71]: SAT is NP-complete
Searching is as easy as checking if and only if it is for SAT.

Relating BOX and SAT

There is a correspondence between BOX and SAT, e.g., between

$\underline{\heartsuit}\quad\underline{\diamondsuit}$	$\heartsuit\quad\underline{\diamondsuit}$	$\underline{\heartsuit}\quad\square$	$\diamondsuit\quad\underline{\square}$

and $(\neg x \vee \neg y) \wedge (x \vee \neg y) \wedge (\neg x \vee z) \wedge (y \vee \neg z)$:

Relating BOX and SAT

There is a correspondence between BOX and SAT, e.g., between

$\underline{\heartsuit}$ $\underline{\diamondsuit}$	$\heartsuit$ $\underline{\diamondsuit}$	$\underline{\heartsuit}$ $\square$	$\diamondsuit$ $\underline{\square}$

and $(\neg x \lor \neg y) \land (x \lor \neg y) \land (\neg x \lor z) \land (y \lor \neg z)$:

- x corresponds to $\heartsuit$, $\neg x$ corresponds to $\underline{\heartsuit}$
- y corresponds to $\diamondsuit$, $\neg y$ corresponds to $\underline{\diamondsuit}$
- z corresponds to $\square$, $\neg z$ corresponds to $\underline{\square}$
- red/blue coloring corresponds to assignment of literals to true/false (**1**/**0**)

Relating BOX and SAT

There is a correspondence between BOX and SAT, e.g., between

$\underline{\heartsuit}\ \ \underline{\diamondsuit}$	$\heartsuit\ \ \underline{\diamondsuit}$	$\underline{\heartsuit}\ \ \square$	$\diamondsuit\ \ \underline{\square}$

and $(\neg x \vee \neg y) \wedge (x \vee \neg y) \wedge (\neg x \vee z) \wedge (y \vee \neg z)$:

- x corresponds to $\heartsuit$, $\neg x$ corresponds to $\underline{\heartsuit}$
- y corresponds to $\diamondsuit$, $\neg y$ corresponds to $\underline{\diamondsuit}$
- z corresponds to $\square$, $\neg z$ corresponds to $\underline{\square}$
- red/blue coloring corresponds to assignment of literals to true/false (**1**/**0**)

Note:

Relating BOX and SAT

There is a correspondence between BOX and SAT, e.g., between

$\underline{\heartsuit}$ $\underline{\diamondsuit}$	$\heartsuit$ $\underline{\diamondsuit}$	$\underline{\heartsuit}$ $\square$	$\diamondsuit$ $\underline{\square}$

and $(\neg x \lor \neg y) \land (x \lor \neg y) \land (\neg x \lor z) \land (y \lor \neg z)$:

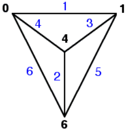
- x corresponds to $\heartsuit$, $\neg x$ corresponds to $\underline{\heartsuit}$
- y corresponds to $\diamondsuit$, $\neg y$ corresponds to $\underline{\diamondsuit}$
- z corresponds to $\square$, $\neg z$ corresponds to $\underline{\square}$
- red/blue coloring corresponds to assignment of literals to true/false (**1**/**0**)

Note:

- assignment of variables gives values of all literals
- if we can solve SAT, we can solve BOX (and vice versa)

Practical Applications of SAT Solving

formal verification

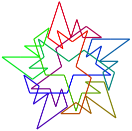


graph theory

bioinformatics

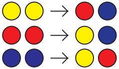
train safety

planning



combinatorics

cryptography



rewrite termination

encode

SAT solver

decode

from `http://www.cs.utexas.edu/users/marijn/talks/Ptn-Linz.pdf`

JKU

Logics in this Lecture

In this lecture, we consider different logic-based languages:

- propositional logic (SAT)
  - simple language: only atoms and connectives
  - low expressiveness, low complexity
  - very successful in industry (e.g., verification)
- first-order logic (predicate logic)
  - rich language: predicates, functions, terms, quantifiers
  - great power of expressiveness, high complexity
  - many applications in mathematics and verification
- satisfiability modulo theories (SMT)
  - customizable language: user decides
  - as much expressiveness as required
    as much complexity as necessary
  - very popular and successful in industry

Logic-Based Languages (Logics)

- A logic consists of
  - a set of symbols (like $\vee, \wedge, \neg, \top, \bot, \forall, \exists \ldots$)
  - a set of variables (like $x, y, z, \ldots$)
  - concise syntax: well-formedness of expressions
  - concise semantics: meaning of expressions
- Logics support reasoning for
  - derivation of “new” knowledge
  - proving the truth/falsity of a statement (satisfiability checking)
- Different logics differ in their
  - truth values: binary (true, false), multi-valued (true, false, unknown), fuzzy (between 0 and 1, e.g., $[0, 1]$ as subset of the real numbers)
  - expressiveness (what can be formulated in the logic?)
  - complexity (how expensive is reasoning?)