FIRST-ORDER LOGIC

Semantics

Wolfgang Schreiner and Wolfgang Windsteiger
Wolfgang.(Schreiner|Windsteiger)@risc.jku.at
Research Institute for Symbolic Computation (RISC)
Johannes Kepler University (JKU), Linz, Austria
http://www.risc.jku.at
The Semantics of First-Order Logic

In first-order logic, the semantics (meaning) depends on a structure and an assignment.

- **A structure** \((D, I)\) consists of a domain \(D\) and an interpretation \(I\) on \(D\):
  - A domain is a non-empty collection of objects (e.g., a set \(D \neq \emptyset\)).
    - The “universe” about which a first-order logic formula talks.
  - An interpretation maps every constant and function/predicate symbol to its meaning:
    - Constant \(c \in C\): \(I(c)\) is an object in \(D\) \(I(c) \in D\).
    - Function symbol \(f \in F\) of arity \(n\): \(I(f)\) is an \(n\)-ary function on \(D\) \(I(f): D^n \rightarrow D\).
    - Predicate symbol \(p \in P\) of arity \(n\): \(I(p)\) is an \(n\)-ary predicate/relation on \(D\) \(I(p) \subseteq D^n\).

- **An assignment** \(a\) maps every variable to its meaning:
  - Variable \(v \in V\): \(a(v)\) is an object in \(D\) \(a(v) \in D\).

\[
\begin{align*}
D &= \mathbb{N} \\
I &= [0 \mapsto \text{zero}, + \mapsto \text{add}, < \mapsto \text{less-than}, \ldots] \\
a &= [x \mapsto \text{one}, y \mapsto \text{zero}, z \mapsto \text{three}, \ldots]
\end{align*}
\]
Informal Semantics

Terms  The meaning of a term is an object in $D$.

- The meaning of a variable $v$ is the object assigned to it by $a$, i.e., $a(v)$.
- The meaning of a constant $c$ is its interpretation in $I$, i.e., $I(c)$.
- The meaning of a function application $f(t_1, \ldots, t_n)$ is the result of applying its interpretation $I(f)$ to the meanings of $t_1, \ldots, t_n$.

Formulas  The meaning of a formula is “true” or “false”.

- The meaning of an atomic formula $p(t_1, \ldots, t_n)$ is the result of applying its interpretation $I(p)$ to the meanings of $t_1, \ldots, t_n$.
  - An equality $t_1 = t_2$ is “true”, if $t_1$ has the same meaning as $t_2$.
- The meaning of the propositional constructions is as already known.
- $(\forall x : F)$ is true if $F$ is true for all possible objects assigned to $x$ in $a$.
- $(\exists x : F)$ is true if $F$ is true for some possible object assigned to $x$ in $a$. 
The Formal Semantics of Terms

\[(D, I) \quad \overrightarrow{a} \quad \llbracket t \rrbracket \quad d \in D\]

- **Term semantics** \(\llbracket t \rrbracket_a^{D,I} \in D\)
  - Given structure \((D, I)\) and assignment \(a\), the semantics of term \(t\) is an object in \(D\).
    \[t ::= v \mid c \mid f(t_1, \ldots, t_n)\]
  - The meaning of a **variable** is the value given by the assignment:
    \[\llbracket v \rrbracket_a^{D,I} := a(v)\]
  - The meaning of a **constant** is the value given by the interpretation:
    \[\llbracket c \rrbracket_a^{D,I} := I(c)\]
  - The meaning of a **function application** is the result of the interpretation of the function symbol applied to the values of the argument terms:
    \[\llbracket f(t_1, \ldots, t_n) \rrbracket_a^{D,I} := I(f)(\llbracket t_1 \rrbracket_a^{D,I}, \ldots, \llbracket t_n \rrbracket_a^{D,I})\]

The recursive definition of a function evaluating a term.
Example

\[ D = \mathbb{N} = \{\text{zero, one, two, three, \ldots}\} \]

\[ I = [0 \mapsto \text{zero, } + \mapsto \text{add, \ldots}] \]

\[ a = [x \mapsto \text{one, } y \mapsto \text{two, \ldots}] \]

\[
\left[ x + (y + 0) \right]_a^{D,I} = \text{add}(\left[ x \right]_a^{D,I}, \left[ y + 0 \right]_a^{D,I})
\]
\[
= \text{add}(a(x), \left[ y + 0 \right]_a^{D,I})
\]
\[
= \text{add}(\text{one}, \left[ y + 0 \right]_a^{D,I})
\]
\[
= \text{add}(\text{one}, \text{add}(\left[ y \right]_a^{D,I}, \left[ 0 \right]_a^{D,I}))
\]
\[
= \text{add}(\text{one}, \text{add}(a(y), I(0)))
\]
\[
= \text{add}(\text{one}, \text{add}(\text{two}, \text{zero}))
\]
\[
= \text{add}(\text{one}, \text{two})
\]
\[
= \text{three}.
\]

The meaning of the term with the “usual” interpretation.

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Example

\[ D = \mathcal{P}(\mathbb{N}) = \{\emptyset, \{\text{zero}\}, \{\text{one}\}, \{\text{two}\}, \ldots, \{\text{zero, one}\}, \ldots\} \]

\[ I = \{0 \mapsto \emptyset, + \mapsto \text{union}, \ldots\} \]

\[ a = \{x \mapsto \{\text{one}\}, y \mapsto \{\text{two}\}, \ldots\} \]

\[
\left\lfloor x + (y + 0) \right\rfloor_{a}^{D,I} = \text{union}(\left\lfloor x \right\rfloor_{a}^{D,I}, \left\lfloor y + 0 \right\rfloor_{a}^{D,I})
\]

\[ = \text{union}(a(x), \left\lfloor y + 0 \right\rfloor_{a}^{D,I}) \]

\[ = \text{union}(\{\text{one}\}, \left\lfloor y + 0 \right\rfloor_{a}^{D,I}) \]

\[ = \text{union}(\{\text{one}\}, \text{union}(\left\lfloor y \right\rfloor_{a}^{D,I}, \left\lfloor 0 \right\rfloor_{a}^{D,I})) \]

\[ = \text{union}(\{\text{one}\}, \text{union}(a(y), I(0))) \]

\[ = \text{union}(\{\text{one}\}, \text{union}(\{\text{two}\}, \emptyset)) \]

\[ = \text{union}(\{\text{one}\}, \{\text{two}\}) \]

\[ = \{\text{one, two}\} \]

The meaning of the term with another interpretation.
The Formal Semantics of Formulas

\[(D, I) \xrightarrow{a} [F] \rightarrow \text{true, false}\]

- **Formula semantics** \([F]_a^{D,I} \in \{\text{true, false}\}\)
  - Given structure \((D, I)\) and assignment \(a\), the semantics of formula \(F\) is a truth value.
    \[F ::= p(t_1, \ldots, t_n) \mid \top \mid \bot \mid \ldots \mid (\forall v : F) \mid (\exists v : F)\]
  - The meaning of an **atomic formula** is the result of the interpretation of the predicate symbol applied to the values of the argument terms (fixed interpretation of equality).
    \[[p(t_1, \ldots, t_n)]_a^{D,I} := I(p)([t_1]_a^{D,I}, \ldots, [t_n]_a^{D,I})\]
    \[[t_1 = t_2]_a^{D,I} := \begin{cases} \text{true} & \text{if } [t_1]_a^{D,I} = [t_2]_a^{D,I} \\ \text{false} & \text{else} \end{cases}\]
  - The meaning of the **logical constants**:
    \[[\top]_a^{D,I} := \text{true} \quad [\bot]_a^{D,I} := \text{false}\]

The meaning of the basic formulas.
The Semantics of Propositional Formulas

The meaning of the logical connectives:

$$\left[ \neg F \right]_{a}^{D,I} := \begin{cases} \text{true} & \text{if } \left[ F \right]_{a}^{D,I} = \text{false} \\ \text{false} & \text{else} \end{cases}$$

$$\left[ F_1 \land F_2 \right]_{a}^{D,I} := \begin{cases} \text{true} & \text{if } \left[ F_1 \right]_{a}^{D,I} = \left[ F_2 \right]_{a}^{D,I} = \text{true} \\ \text{false} & \text{else} \end{cases}$$

$$\left[ F_1 \lor F_2 \right]_{a}^{D,I} := \begin{cases} \text{false} & \text{if } \left[ F_1 \right]_{a}^{D,I} = \left[ F_2 \right]_{a}^{D,I} = \text{false} \\ \text{true} & \text{else} \end{cases}$$

$$\left[ F_1 \rightarrow F_2 \right]_{a}^{D,I} := \begin{cases} \text{false} & \text{if } \left[ F_1 \right]_{a}^{D,I} = \text{true} \text{ and } \left[ F_2 \right]_{a}^{D,I} = \text{false} \\ \text{true} & \text{else} \end{cases}$$

$$\left[ F_1 \leftrightarrow F_2 \right]_{a}^{D,I} := \begin{cases} \text{true} & \text{if } \left[ F_1 \right]_{a}^{D,I} = \left[ F_2 \right]_{a}^{D,I} \\ \text{false} & \text{else} \end{cases}$$

An embedding of the semantics of propositional logic into first-order logic.
The Semantics of Quantified Formulas

■ \((\forall x: F)\) is true, if \(F\) is true for every possible object \(d\) assigned to variable \(x\):

\[
[\forall x: F]_{a}^{D,I} := \begin{cases} 
\text{true} & \text{if } [F]_{a[x \mapsto d]}^{D,I} = \text{true for all } d \text{ in } D \\
\text{false} & \text{else}
\end{cases}
\]

■ \((\exists x: F)\) is true, if \(F\) is true for at least one possible object \(d\) assigned to variable \(x\):

\[
[\exists x: F]_{a}^{D,I} := \begin{cases} 
\text{true} & \text{if } [F]_{a[x \mapsto d]}^{D,I} = \text{true for some } d \text{ in } D \\
\text{false} & \text{else}
\end{cases}
\]

■ Assignment \(a\) updated by the assignment of object \(d\) to variable \(x\):

\[
a[x \mapsto d](y) = \begin{cases} 
d & \text{if } x = y \\
a(y) & \text{else}
\end{cases}
\]

The core of the semantics of first-order logic.
Example

\[ D = \mathbb{N}_3 = \{ \text{zero}, \text{one}, \text{two} \} \quad I = [0 \mapsto \text{zero}, + \mapsto \text{add}, \ldots] \quad a = [x \mapsto \text{one}, y \mapsto \text{two}, z \mapsto \text{two}, \ldots] \]

\( \forall x \colon \exists y : x + y = z \)\(^{D,I}_a = \) ?

- \( \exists y : x + y = z \)\(^{D,I}_{a[x \mapsto \text{zero}]} = \) true
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{zero}, y \mapsto \text{zero}]} = \) false
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{zero}, y \mapsto \text{one}]} = \) false
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{zero}, y \mapsto \text{two}]} = \) true

- \( \exists y : x + y = z \)\(^{D,I}_{a[x \mapsto \text{one}]} = \) true
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{one}, y \mapsto \text{zero}]} = \) false
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{one}, y \mapsto \text{one}]} = \) true
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{one}, y \mapsto \text{two}]} = \) false

- \( \exists y : x + y = z \)\(^{D,I}_{a[x \mapsto \text{two}]} = \) true
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{two}, y \mapsto \text{zero}]} = \) true
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{two}, y \mapsto \text{one}]} = \) false
  - \( x + y = z \)\(^{D,I}_{a[x \mapsto \text{two}, y \mapsto \text{two}]} = \) false

\( \forall x : \exists y : x + y = z \)\(^{D,I}_a = \) true.
Semantics: Structures and Assignments

- \( \forall n: R(n,n) \)
  - The domain of natural numbers with \( R \) interpreted as the divisibility relation.
  - “Every natural number is divisible by itself”: true (for every assignment).

- \( \forall n: R(n,n) \)
  - The domain of natural numbers with \( R \) interpreted as the less-than relation.
  - “Every natural number is less than itself”: false (for every assignment).

- \( \exists x: R(y,x) \land R(x,z) \)
  - The domain of natural numbers with \( R \) interpreted as the less-than relation.
  - “There exists a natural number \( x \) with \( y < x \) and \( x < z \).”
  - Assignment \([y \mapsto 2, z \mapsto 4]\): true (there is the value \( x = 4 \) with \( 2 < x \) and \( x < 4 \)).
  - Assignment \([y \mapsto 2, z \mapsto 3]\): false (there is no value for \( x \) with \( 2 < x \) and \( x < 3 \)).

The truth value of a formula depends on the structure and the assignment.
Semantics: Nested Quantifiers

Consider the domain of natural numbers with the usual interpretation of <.

■ $(\forall x: \exists y: x < y)$: true.

  □ “For every natural number $x$ there exists some $y$ such that $x$ is less than $y$”.
  □ For every natural number $x$, there is indeed such a $y$, namely $y := x + 1$.

■ $(\exists y: \forall x: x < y)$: false

  □ “There exists some natural number $y$ such that every $x$ is less than $y$.”
  □ We assume that the formula is true and derive a contradiction. Because of the assumption, there exists some natural number $y$ such that $(\forall x: x < y)$ is true. But then, since $x < y$ is true for every value of $x$, it is also true for $x := y$. Thus $y < y$ is true, which we know to be false.

The order of nested quantifiers matters.
Semantic Notions: Satisfiability and Validity

Let $F$ denote a formula, $M = (D, I)$ a structure, $a$ an assignment.

**Satisfiability**  Formula $F$ is **satisfiable**, if there exists some structure $M$ and assignment $a$ such that $\models^M_a F = \text{true}$.

- Example: $p(0, x)$ is satisfiable; $q(x) \land \neg q(x)$ is not.

**Model**  Structure $M$ is a **model** of formula $F$, written as $M \models F$, if for every assignment $a$, we have $\models^M_a F = \text{true}$.

- Example: $(\mathbb{N}, [0 \mapsto \text{zero}, p \mapsto \text{less-equal}]) \models p(0, x)$

**Validity**  Formula $F$ is **valid**, written as $\models F$, if every structure $M$ is a model of $F$, i.e., for every structure $M$ we have $M \models F$.

- Example: $\models p(x) \land (p(x) \rightarrow q(x)) \rightarrow q(x)$
Semantic Notions: Logical Consequence and Equivalence

**Logical Consequence** Formula $F_2$ is a logical consequence of formula $F_1$, written as $F_1 \models F_2$, if for every structure $M$ and assignment $a$, the following is true:

If $\llbracket F_1 \rrbracket^M_a = \text{true}$, then also $\llbracket F_2 \rrbracket^M_a = \text{true}$.

- Example: $p(x) \land (p(x) \land q(x)) \models q(x)$

**Logical Consequence Generalized** Formula $F$ is a logical consequence of formulas $F_1, \ldots, F_n$, written $F_1, \ldots, F_n \models F$, if for every $M$ and $a$ the following is true:

If for every formula $F_i$ we have $\llbracket F_i \rrbracket^M_a = \text{true}$, then $\llbracket F \rrbracket^M_a = \text{true}$.

- Example: $p(x), q(x) \models p(x) \land q(x)$

**Logical Equivalence** Formulas $F_1$ and $F_2$ are logically equivalent, written as $F_1 \iff F_2$, if and only if $F_1$ is a logical consequence of $F_2$ and vice versa, i.e., $F_1 \models F_2$ and $F_2 \models F_1$.

- Example: $p(x) \rightarrow q(x) \iff \neg p(x) \lor q(x)$
Semantic Notions: Propositions

Satisfiability and Validity

- $F$ is satisfiable, if $\neg F$ is not valid.
- $F$ is valid, if $\neg F$ is not satisfiable.

Logical Consequence and Equivalence

- Formula $F_2$ is a logical consequence of formula $F_1$ (i.e., $F_1 \models F_2$) if and only if the formula $(F_1 \rightarrow F_2)$ is valid.
- Formula $F$ is a logical consequence of formulas $F_1, \ldots, F_n$ (i.e., $F_1, \ldots, F_n \models F$) if and only if the formula $(F_1 \land \ldots \land F_n \rightarrow F)$ is valid.
- Formula $F_1$ and formula $F_2$ are logically equivalent (i.e., $F_1 \iff F_2$) if and only if the formula $(F_1 \leftrightarrow F_2)$ is valid.

Logical consequence/equivalence reduced to validity of an implication/equivalence.
Logical Equivalence: Formula Substitutions

Assume $F \iff F'$ and $G \iff G'$. Then we have the following equivalences:

- $\neg F \iff \neg F'$
- $F \land G \iff F' \land G'$
- $F \lor G \iff F' \lor G'$
- $F \to G \iff F' \to G'$
- $F \iff G \iff F' \iff G'$
- $\forall x: F \iff \forall x: F'$
- $\exists x: F \iff \exists x: F'$

Logically equivalent formulas can be substituted in any context.
Logical Equivalence: Rules

In addition to the logical equivalences for connectives in propositional logic:

\[ \neg \forall x: F \iff \exists x: \neg F \]  \hspace{1cm} \text{(De Morgan’s Law)}

\[ \neg \exists x: F \iff \forall x: \neg F \]  \hspace{1cm} \text{(De Morgan’s Law)}

\[ \forall x: (F_1 \land F_2) \iff (\forall x: F_1) \land (\forall x: F_2) \]

\[ \exists x: (F_1 \lor F_2) \iff (\exists x: F_1) \lor (\exists x: F_2) \]

\[ \forall x: (F_1 \lor F_2) \iff F_1 \lor (\forall x: F_2) \] \hspace{1cm} \text{if } x \text{ is not free in } F_1

\[ \exists x: (F_1 \land F_2) \iff F_1 \land (\exists x: F_2) \] \hspace{1cm} \text{if } x \text{ is not free in } F_1

For a finite domain \( \{v_1, \ldots, v_n\} \):

\[ \forall x: F \iff F[v_1/x] \land \ldots \land F[v_n/x] \]

\[ \exists x: F \iff F[v_1/x] \lor \ldots \lor F[v_n/x] \]
Logical Equivalence: Examples

Push negations from the outside to the inside:
\[
\neg (\forall x: p(x) \rightarrow \exists y: q(x,y)) \iff \exists x: (\neg(p(x) \rightarrow \exists y: q(x,y))
\]
\[
\iff \exists x: (\neg p(x) \lor \exists y: q(x,y))
\]
\[
\iff \exists x: (((\neg p(x)) \land \neg \exists y: q(x,y))
\]
\[
\iff \exists x: (p(x) \land \exists y: q(x,y))
\]
\[
\iff \exists x: (p(x) \land \forall y: \neg q(x,y))
\]

Reduce the scope of quantifiers:
\[
\forall x, y: (p(x) \rightarrow q(x,y)) \iff \forall x, y: (\neg p(x) \lor q(x,y))
\]
\[
\iff \forall x: (\neg p(x) \lor \forall y: q(x,y))
\]
\[
\iff \forall x: (p(x) \rightarrow \forall y: q(x,y))
\]

Replace quantification in a finite domain \(D = \{0, 1, 2\}::
\[
\forall x: p(x) \iff p(0) \land p(1) \land p(2)
\]