Propositional Skeleton

Example (arbitrary LRA formula)

\[ x \neq y \land (2 \times x \leq z \lor \neg(x - y \geq z \land z \leq y)) \]

eliminate \( \neq \) by disjunction

\[ (x < y \lor x > y) \land (2 \times x \leq z \lor \neg(x - y \geq z \land z \leq y)) \]

which is abstracted to a propositional formula called “propositional skeleton”

\[ (a \lor b) \land (c \lor \neg(d \land e)) \]

with \( \alpha(x < y) = a, \alpha(x > y) = b, \ldots \)

SAT solver enumerates solutions, e.g., \( a = b = c = d = e = 1 \)

check solution literals with theory solver, e.g., Fourier-Motzkin

spurious solutions (disproven by theory solver) added as “lemma”, e.g., \( \neg(a \land b \land c \land c \land d \land e) \)

or just \( \neg(a \land b) \) after minimization

continue until SAT solver says unsatisfiable or theory solver satisfiable
Lemmas on Demand

this is an extremely “lazy” version of DPLL (T) / CDCL(T)

LemmasOnDemand(φ)

ψ = PropositionalSkeleton(φ)

let α be the abstraction function, mapping theory literals to prop. literals

while ψ has satisfiable assignment σ

let l₁, . . . , lₙ be all the theory literals with σ(α(lᵢ)) = 1

check conjunction L = l₁ ∧ · · · ∧ lₙ with theory solver

if theory solver returns satisfying assignment ρ return satisfiable

determine “small” sub-set \{k₁, . . . , kₘ\} ⊆ \{l₁, . . . , lₙ\} where \n \nK = k₁ ∧ · · · ∧ kₘ remains unsatisfiable (by theory solver)

add lemma ¬K to ψ, actually replace ψ by ψ ∧ α(¬K)

return unsatisfiable

note that these lemmas ¬K are all clauses
Minimal Unsatisfiable Set (MUS)

motivation: the lemmas we add in “lemmas on demand” should be small

\[
\text{MUS} = (a \lor \neg b) \land (a \lor b) \land (\neg a \lor \neg c) \land (\neg a \lor c) \land (a \lor \neg c) \land (a \lor c)
\]

- given an unsatisfiable set of “constraints” \( S \) (set of literals, or clauses)
- an MUS \( M \) is a sub-set \( M \subseteq S \) such that
  - \( M \) is still unsatisfiable
  - any \( M' \subset M \) (with \( M' \neq M \)) is satisfiable
- so an MUS is a “minimal” inconsistent subset
  - all constraints in the MUS are necessary for \( M \) to be inconsistent
  - so one minimal way to explain inconsistency of \( S \)
- note that “being inconsistent” is a monotone property
  - if \( A \subseteq B \) is a set of constraints
  - if \( A \) is unsatisfiable then \( B \) is unsatisfiable
  - essential for algorithms to compute an MUS
Iterative Destructive Algorithm for MUS Computation

destructive = remove constraints from an over-approximation of an MUS

\[
\text{IterativeDestructiveMUS}(S) \\
M = S \\
D = S \\
\text{while } D \neq \emptyset \\
\text{pick constraint } C \in D \\
\text{if } M \setminus \{C\} \text{ unsatisfiable remove } C \text{ from } M \\
\text{remove } C \text{ from } D \\
\text{return } M
\]

needs exactly \(|S|\) satisfiability checks

any-time algorithm: preliminary result \(M\) remains inconsistent

can stop any time
QuickXplain Variant of MUS Computation

quickly “zoom in” on one MUS (particularly if there is a small one)

$$\text{QuickMUSRecursive}(D)$$

if $M \setminus D$ is satisfiable
  if $|D| > 1$
    let $D = L \cup R$ with $|L|, |R| > 0$
    $\ldots \geq \left\lfloor \frac{|D|}{2} \right\rfloor$
    $\text{QuickMUSRecursive}(L)$
    $\text{QuickMUSRecursive}(R)$

else remove $D$ from $M$

$$\text{QuickMUS}(S)$$

global variable $M = S$

$\text{QuickMUSRecursive}(S)$

return $M$

needs at most $2 \cdot |S|$ and at least $|M|$ satisfiability checks
Theory of Arrays

- functions “read” and “write”: \( \text{read}(a, i), \text{write}(a, i, v) \)

- axioms

  \[
  \forall a, i, j : i = j \rightarrow \text{read}(a, i) = \text{read}(a, j) \quad \text{array congruence}
  \]

  \[
  \forall a, v, i, j : i = j \rightarrow \text{read}(\text{write}(a, i, v), j) = v \quad \text{read over write 1}
  \]

  \[
  \forall a, v, i, j : i \neq j \rightarrow \text{read}(\text{write}(a, i, v), j) = \text{read}(a, j) \quad \text{read over write 2}
  \]

- used to model memory (HW and SW)

- eagerly reduce arrays to uninterpreted functions by eliminating “write”

  \[
  \text{read}(\text{write}(a, i, v), j) \quad \text{replaced by} \quad (i = j ? v : \text{read}(a, j))
  \]

- more sophisticated non-eager algorithms are usually faster

- such as for instance the lemmas-on-demand algorithm in Boolector
Simple Array Example

\[ i \neq j \land u = \text{read}(\text{write}(a, i, v), j) \land v = \text{read}(a, j) \land u \neq v \]

eliminate “write”

\[ i \neq j \land u = (i = j \ ? \ v : \text{read}(a, j)) \land v = \text{read}(a, j) \land u \neq v \]

simplify conditional by assuming “\( i \neq j \)”

\[ i \neq j \land u = \text{read}(a, j) \land v = \text{read}(a, j) \land u \neq v \]

applying congruence for both “read”

\[ i \neq j \land u = \text{read}(a, j) = \text{read}(a, j) = v \land u \neq v \]

which is clearly \textit{unsatisfiable}
More Complex Array Example for Checking Aliasing

<table>
<thead>
<tr>
<th>original</th>
<th>optimized</th>
</tr>
</thead>
<tbody>
<tr>
<td>assert (i != k);</td>
<td>int t = a[k];</td>
</tr>
<tr>
<td>a[i] = a[k];</td>
<td>a[i] = t;</td>
</tr>
<tr>
<td>a[j] = a[k];</td>
<td>a[j] = t;</td>
</tr>
</tbody>
</table>

\[ i \neq k \]
\[ t = \text{read}(a, k) \]
\[ b_1 = \text{write}(a, i, t) \]
\[ c_1 = \text{write}(a, i, t) \]
\[ b_2 = \text{write}(b_1, j, s) \]
\[ c_2 = \text{write}(c_1, j, t) \]
\[ s = \text{read}(b_1, k) \]

original \neq \text{optimized} \quad \text{iff} \quad \exists l \mbox{ with } \text{read}(b_2, l) \neq \text{read}(c_2, l)
Thus, \textit{original} \neq \textit{optimized} iff

\begin{align*}
i \neq k \\
t &= \text{read}(a, k) \\
b_1 &= \text{write}(a, i, t) \\
b_2 &= \text{write}(b_1, j, s) \\
c_1 &= \text{write}(a, i, t) \\
c_2 &= \text{write}(c_1, j, t) \\
s &= \text{read}(b_1, k) \\
\text{read}(b_2, l) \neq \text{read}(c_2, l)
\end{align*}

satisfiable
Aliasing Example Continued 2

thus \textit{original} \neq \textit{optimized} iff

\begin{align*}
i \neq k \\
t &= \text{read}(a, k) \\
b_1 &= \text{write}(a, i, t) \\
b_2 &= \text{write}(b_1, j, s) \\
c_1 &= \text{write}(a, i, t) \\
c_2 &= \text{write}(c_1, j, t) \\
s &= \text{read}(b_1, k) \\
u &= \text{read}(b_2, l) \\
v &= \text{read}(c_2, l) \\
u \neq v
\end{align*}

satisfiable
Aliasing Example Continued 3

after eliminating $c_2$

$$i \neq k$$
$$t = \text{read}(a, k)$$
$$b_1 = \text{write}(a, i, t)$$
$$b_2 = \text{write}(b_1, j, s)$$
$$c_1 = \text{write}(a, i, t)$$
$$c_2 = \text{write}(c_1, j, t)$$
$$s = \text{read}(b_1, k)$$
$$u = \text{read}(b_2, l)$$
$$v = (i = j \ ? \ t : \text{read}(c_1, l))$$
$$u \neq v$$
After eliminating $c_2, c_1$

\[i \neq k\]
\[t = \text{read}(a, k)\]
\[b_1 = \text{write}(a, i, t)\]
\[b_2 = \text{write}(b_1, j, s)\]
\[c_1 = \text{write}(a, i, t)\]
\[c_2 = \text{write}(c_1, j, t)\]
\[s = \text{read}(b_1, k)\]
\[u = \text{read}(b_2, l)\]
\[v = (l = j \ ? \ t : (l = i \ ? \ t : \text{read}(a, l)))\]
\[u \neq v\]
after eliminating $c_2, c_1, b_2$

$i \neq k$
$t = \text{read}(a, k)$
$b_1 = \text{write}(a, i, t)$
$b_2 = \text{write}(b_1, j, s)$
$c_1 = \text{write}(a, i, t)$
$c_2 = \text{write}(c_1, j, t)$
$s = \text{read}(b_1, k)$
$u = (l = j \ ? \ s : \text{read}(b_1, l))$
$v = (l = j \ ? \ t : (l = i \ ? \ t : \text{read}(a, l)))$
$u \neq v$
after eliminating $c_2$, $c_1$, $b_2$, $b_1$

\begin{align*}
i &\neq k \\
t &\leftarrow \text{read}(a, k) \\
b_1 &\leftarrow \text{write}(a, i, t) \\
b_2 &\leftarrow \text{write}(b_1, j, s) \\
c_1 &\leftarrow \text{write}(a, i, t) \\
c_2 &\leftarrow \text{write}(c_1, j, t) \\
s &\leftarrow (k \not= i \ ? \ t : \text{read}(a, k)) \\
u &\leftarrow (l = j \ ? \ s : (l = i \ ? \ t : \text{read}(a, l))) \\
v &\leftarrow (l = j \ ? \ t : (l = i \ ? \ t : \text{read}(a, l))) \\
u &\neq v
\end{align*}
result after “write” elimination

\[ i \neq k \]
\[ t = \text{read}(a, k) \]
\[ s = (k = i \ ? \ t : \text{read}(a, k)) \]
\[ u = (l = j \ ? \ s : (l = i \ ? \ t : \text{read}(a, l))) \]
\[ v = (l = j \ ? \ t : (l = i \ ? \ t : \text{read}(a, l))) \]
\[ u \neq v \]
after eliminating conditionals (if-then-else)

\[
i \neq k
\]
\[
t = \text{read}(a, k)
\]
\[
k = i \rightarrow s = t
\]
\[
k \neq i \rightarrow s = \text{read}(a, k)
\]
\[
l = j \rightarrow u = s
\]
\[
l \neq j \land l = i \rightarrow u = t
\]
\[
l \neq j \land l \neq i \rightarrow u = \text{read}(a, l)
\]
\[
l = j \rightarrow v = t
\]
\[
l \neq j \land l = i \rightarrow v = t
\]
\[
l \neq j \land l \neq i \rightarrow v = \text{read}(a, l)
\]
\[
u \neq v
\]

now treat “read” as uninterpreted function (say \(f\))
check with lemmas-on-demand and congruence closure
Ackermann’s Reduction

formula in theory of uninterpreted functions with equality and disequality:

1. flatten terms by introducing new variables as before
   - remove nested function applications
   - equalities and disequalities have at least one variable on left or right side

2. instantiate congruence axiom in all possible ways:
   - replace all function applications $f(u)$ by new variable $f^u$
   - replace all function applications $f(u, v)$ by new variable $f^{u,v}$ etc.

3. if formula contains $f^u$ and $f^v$ add $u = v \rightarrow f^u = f^v$ as lemma etc.

4. use decision procedure for theory of equality and disequality
   - if the resulting formula after the first two steps contains $n$ variables
   - then only need to consider domains with $n$ elements
   - or bit-vectors of length $\lceil \log_2 n \rceil$ bits
   - allows eager encoding into SAT

“eagerly” generates all instantiations of the congruence axioms as lemmas
Example of Ackermann’s Reduction

we start with an already flattened formula

\[ x = f(y) \land y = f(x) \land x \neq y \]

after second step

\[ x = f^y \land y = f^x \land x \neq y \]

after adding lemmas in second step

\[ x = f^y \land y = f^x \land x \neq y \land (x = y \rightarrow f^x = f^y) \]

resulting formula has 4 variables thus needs bit-vectors of length 2
Example of Ackermann’s Reduction to Bit-Vectors

$ cat ack.smt2
(set-logic QF_BV)
(declare-fun x () (_ BitVec 2))
(declare-fun y () (_ BitVec 2))
(declare-fun fx () (_ BitVec 2))
(declare-fun fy () (_ BitVec 2))
(assert (and (= x fy) (= y fx) (distinct x y) (=> (= x y) (= fx fy))))
(check-sat)
(exit)
$ boolector ack.smt2 -m -d
sat
x 0
y 3
fx 3
fy 0
Theory of Bit-Vectors

- allows “bit-precise” reasoning
  - captures semantics of low-level languages like assembler, C, C++, ...
  - Java / C# also use two-complement representations for int
  - modelling of hardware / circuits on the word-level (RTL)
  - important for security applications and precise test case generation

- many operations
  - logical operations, bit-wise operations (and, or)
  - equalities, inequalities, disequalities
  - shift, concatenation, slicing
  - addition, multiplication, division, modulo, ...

- main approach is reduction to SAT through bit-blasting
  - reduction of bit-vector operations similar to circuit synthesis
  - Ackermann’s Reduction only needs equality and disequality
Bit-Blasting Bit-Vector Equality

for each bit-vector equality $u = v$ with $u$ and $v$ bit-vectors of width $w$

introduce new propositional variables for individual bits

$u_1, \ldots, u_w \quad v_1, \ldots, v_w$

replace $u = v$ by new propositional variable $e_{u=v}$

add the propositional constraint

$e_{u=v} \iff \bigwedge_{i=1}^{w} (u_i \leftrightarrow v_i)$

disequality $u \neq v$ is replaced by $\neg e_{u=v}$

resulting formula satisfiable iff original formula satisfiable
Bit-Blasting Ackermann Example

\[ x = f^y \land y = f^x \land x \neq y \land (x = y \rightarrow f^x = f^y) \]

now replacing the bit-vector equalities and the disequality by new \( e \) variables

\[ e_{x=y} \land e_{y=f^x} \land \neg e_{x=y} \land (e_{x=y} \rightarrow e_{f^x=f^y}) \]

and adding the equality constraints

\[ e_{x=f^y} \iff (x_1 \iff f_1^y) \land (x_2 \iff f_2^y) \]
\[ e_{y=f^x} \iff (y_1 \iff f_1^x) \land (y_2 \iff f_2^x) \]
\[ e_{x=y} \iff (x_1 \iff y_1) \land (x_2 \iff y_2) \]
\[ e_{f^x=f^y} \iff (f_1^x \iff f_1^y) \land (f_2^x \iff f_2^y) \]

gives an “equi-satisfiable” formula which can be checked by SAT solver
Bit-Blasting Ackermann Example in Limboole Syntax

$ cat ackbitblasted.limboole
exfy & eyfx & !exy & (exy -> efxfy) &
(exfy <-> (x1 <-> fy1) & (x2 <-> fy2)) &
(eyfx <-> (y1 <-> fx1) & (y2 <-> fx2)) &
(exy <-> (x1 <-> y1) & (x2 <-> y2)) &
(efxfy <-> (fx1 <-> fy1) & (fx2 <-> fy2))

$ limboole ackbitblasted.limboole -s|grep -v SAT|sort

efxfy = 0
exfy = 1
exy = 0
eyfx = 1
fx1 = 0
fx2 = 1
fy1 = 1
fy2 = 1
x1 = 1
x2 = 1
y1 = 0
y2 = 1