Lemmas on Demand for Lambdas

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Theory of arrays [McCarthy, 1962]

A1 $i = j \rightarrow read(a, i) = read(a, j)$ *(array congruence)*

A2 $i = j \rightarrow read(write(a, i, e), j) = e$ *(read-over-write 1)*

A3 $i \neq j \rightarrow read(write(a, i, e), j) = read(a, j)$ *(read-over-write 2)*

Limitations

- array operations restricted to single array indices
- no efficient modeling of parallel array updates (e.g.: *memset*, *memcpy*)

$\rightarrow$ [Bryant et al., 2002] tackle limitations by using restricted λ-terms in UCLID

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Lambdas as arrays

- *write*(a, i, e):
  $\lambda j \,.\, ite(i = j, e, read(a, j))$
- *memset*(a, i, n, e):
  $\lambda j \,.\, ite(i \leq j \wedge j < i + n, e, read(a, j))$
- *memcpy*(a, b, i, k, n):
  $\lambda j \,.\, ite(k \leq j \wedge j < k + n, read(a, i + j - k), read(b, j))$
- ...

Further applications

- ordered data structures
- arbitrary functions
- SMT-LIB v2 macros
- ...

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can be **symbolic**

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UCLID [Seshia, 2005]

- SMT solver using eager approach
- non-recursive λ-terms
- λ-terms used for modeling arrays and array operations (and more)

Lazy SMT solvers with lambda support:

- CVC4 [Barrett et al., 2011]
- Yices [Dutertre and de Moura, 2006]

Lambda handling in SMT solvers

- λ-terms treated as C-style macros
- eager elimination with β-reduction
- may result in an **exponential blow-up** in formula size

Example [Seshia, 2005]

$$
\begin{aligned}
F &:= P(L_1(a)) \\
L_1 &:= \lambda x \,.\, f_1(L_2(x), L_2(g_1(x))) \\
L_2 &:= \lambda x \,.\, f_2(L_3(x), L_3(g_2(x))) \\
&\vdots \\
L_{k-1} &:= \lambda x \,.\, f_{k-1}(L_k(x), L_k(g_{k-1}(x))) \\
L_k &:= \lambda x \,.\, g_k(x)
\end{aligned}
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2^k instantiations of L_k

$\rightarrow$ avoid with **lazy lambda handling**

Eager Lambda Elimination Worst-Case

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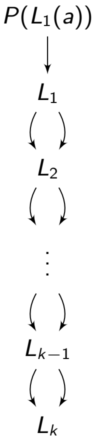

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$$
\begin{array}{c}
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\downarrow \\
L_1
\end{array}
$$

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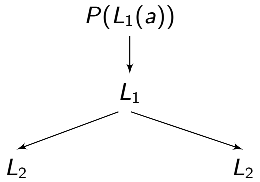

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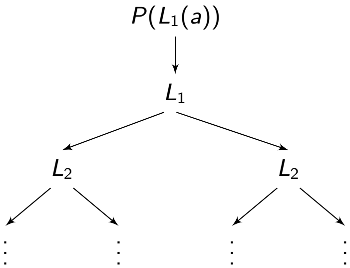

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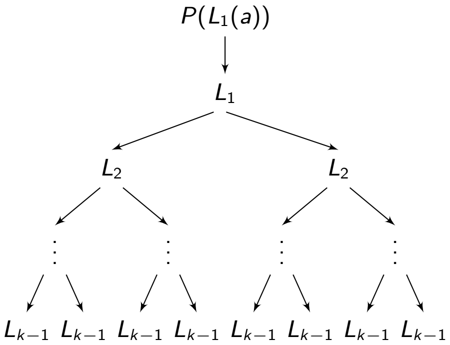

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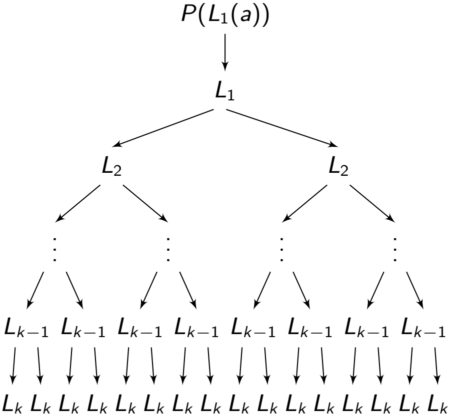

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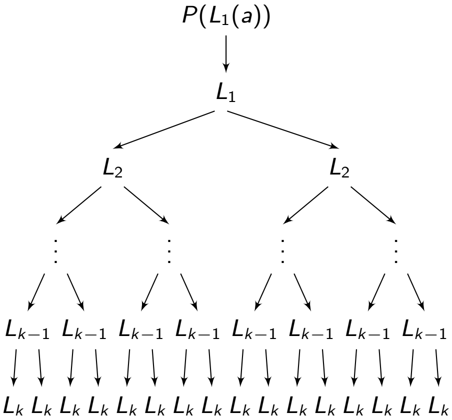

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Boolector

- lazy SMT solver
- employs **lemmas on demand**
- supported theories:
  - fixed size bit vectors
  - arrays
- no quantifiers

Old version (pre-lambda)

- extensionality on arrays

New version

- λ-term support
- extensionality not supported (yet)

Extensionality on arrays

$$a = b$$
$$\Leftrightarrow$$
$$\forall i \,.\, read(a, i) = read(b, i)$$

Extensionality on lambdas

$$\lambda \bar{x} \,.\, \phi = \lambda \bar{y} \,.\, \psi$$
$$\Leftrightarrow$$
$$\forall \bar{a} \,.\, (\lambda \bar{x} \,.\, \phi)(\bar{a}) = (\lambda \bar{y} \,.\, \psi)(\bar{a})$$

Quantifiers with extensionality on lambdas

$$\lambda x \,.\, p(x) = \lambda x \,.\, \top$$
$$\Leftrightarrow$$
$$\forall x \,.\, p(x)$$

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$$\lambda x \;.\; p(x) = \lambda x \;.\; \top$$
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Restrictions

- non-recursive
- non-extensional
- non-higher order functions

Lambdas in Boolector

- arrays represented as λ-terms\uninterpreted functions
  $\rightarrow$ no terms of sort array
  $\rightarrow$ uniform handling of arrays and λ-terms
- SMT-LIB v2 macros treated as curried λ-terms
- **lazy instantiation** of λ-terms
  $\rightarrow$ optional eager elimination
- new decision procedure DP_λ for λ-terms
  $\rightarrow$ generalization of array decision procedure [Brummayer and Biere, 2009]

Lambdas in Boolector

β-reduction Approaches

Partial β-reduction

- like β-reduction in λ-calculus
- λ-terms are expanded "function-wise"
- required for consistency checking in DP$_\lambda$
  $\rightarrow$ considers current assignment

Full β-reduction

- eager elimination of λ-terms
- optional rewriting step

Given a DAG representing a λ-term ...

1. perform DFS post-order traversal
2. consecutively assign arguments to parameters
3. rebuild terms with arguments instead of parameters

Our notation for partial β-reduction: $\lambda_{\bar{x}}[x_1 \backslash a_1, \ldots, x_n \backslash a_n]_{\mathbf{p}}$

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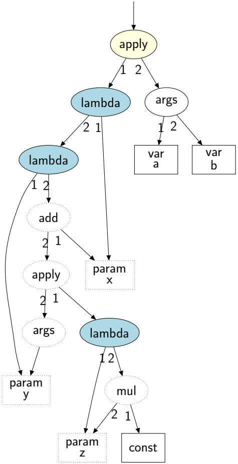

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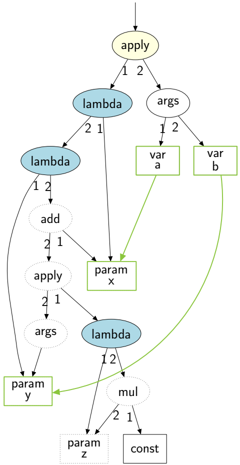

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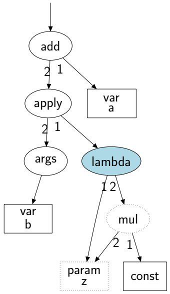
Result partial β-reduction



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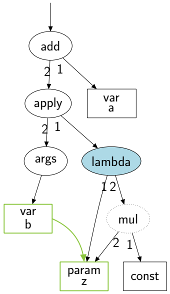

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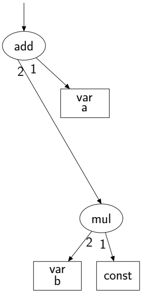

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Refinement loop

1. abstract input formula ϕ (bit vector skeleton)
   $\rightarrow$ introduce **fresh** bit vector variable for each function application
   $\rightarrow$ translate bit vector skeleton into prop. formula

2. let SAT solver "guess" a solution
   $\rightarrow$ if SAT solver returns unsatisfiable, terminate with **unsatisfiable**

3. check if satisfying assignment is **consistent w.r.t.** ϕ ($consistent_\lambda$)
   $\rightarrow$ if check succeeds, terminate with **satisfiable**

4. if check fails, add lemma to **refine** formula abstraction ($lemma_\lambda$)

5. continue with 2

What to check?
Check whether current assignment σ is spurious or not

Rules

- rule C: function congruence axiom EUF

$$\forall \bar{a}, \bar{b}. \bigwedge_{i=1}^{n} \sigma(a_i) = \sigma(b_i) \rightarrow \sigma(f(\bar{a})) = \sigma(f(\bar{b}))$$

$$\ldots \rightarrow \sigma(v_{f(\bar{a})}) = \sigma(v_{f(\bar{b})})$$

- rule B: abstraction variable consistency

$$\sigma(v_{\lambda_{\bar{x}}(a)}) = \sigma(\lambda_{\bar{x}}[x_1 \backslash a_1, \ldots, x_n \backslash a_n]_{\mathbf{p}})$$

$\rightarrow$ Optimization: rule P (see paper for more details)

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Algorithm *consistent*$_\lambda$

- adaption of propagation algorithm in [Brummayer and Biere, 2009]
- associate each function application with resp. function
  $\rightarrow$ maintain hash table ρ for every function
- for each **pair** of function applications in ρ check rule C
- for each function application in ρ check rule B (λ-terms only)
- if a **conflict** occurs, generate a lemma (*lemma*$_\lambda$)
- otherwise, current assignment σ is **valid**

Violation of rule C

$s := g(a_1, \ldots, a_n),\ t := h(b_1, \ldots, b_n) \in \rho(f)$ violate rule C

1. find propagation path p^s (p^t) from s (t) to f
2. collect all *ite* conditions $c_0^s, \ldots, c_j^s$ $(c_0^t, \ldots, c_l^t)$ on path p^s (p^t) that were $\top$ under given assignment σ
3. collect all *ite* conditions $c_0^s, \ldots, c_k^s$ $(c_0^t, \ldots, c_m^t)$ on path p^s (p^t) that were $\bot$ under given assignment σ

Lemma

$$\bigwedge_{i=0}^{j} c_i^s \wedge \bigwedge_{i=0}^{k} \neg c_i^s \wedge \bigwedge_{i=0}^{l} c_i^t \wedge \bigwedge_{i=0}^{m} \neg c_i^t \wedge \bigwedge_{i=0}^{n} a_i = b_i \rightarrow s = t$$

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$$\bigwedge_{i=0}^{j} c_i^s \wedge \bigwedge_{i=0}^{k} \neg c_i^s \wedge \underbrace{\bigwedge_{i=0}^{l} c_i^t \wedge \bigwedge_{i=0}^{m} \neg c_i^t}_{\text{Prop. conditions t}} \wedge \bigwedge_{i=0}^{n} a_i = b_i \rightarrow s = t$$

Violation of rule C

$s := g(a_1, \ldots, a_n),\ t := h(b_1, \ldots, b_n) \in \rho(f)$ violate rule C

1. find propagation path p^s (p^t) from s (t) to f
2. collect all *ite* conditions $c_0^s, \ldots, c_j^s$ $(c_0^t, \ldots, c_l^t)$ on path p^s (p^t) that were $\top$ under given assignment σ
3. collect all *ite* conditions $c_0^s, \ldots, c_k^s$ $(c_0^t, \ldots, c_m^t)$ on path p^s (p^t) that were $\bot$ under given assignment σ

Lemma

$$\bigwedge_{i=0}^{j} c_i^s \wedge \bigwedge_{i=0}^{k} \neg c_i^s \wedge \bigwedge_{i=0}^{l} c_i^t \wedge \bigwedge_{i=0}^{m} \neg c_i^t \wedge \underbrace{\bigwedge_{i=0}^{n} a_i = b_i \rightarrow s = t}_{\text{function congruence}}$$

Violation of rule B

$s := \lambda_{\bar{y}}(a_1, \ldots, a_n) \in \rho(\lambda_{\bar{x}})$, $t := \lambda_{\bar{x}}[x_1 \backslash a_1, \ldots, x_n \backslash a_n]_{\mathbf{p}}$ violates rule B

1. collect conditions $c_0^s, \ldots, c_j^s$, $c_0^s, \ldots, c_k^s$ as before
2. collect all *ite* conditions $c_0^t, \ldots, c_l^t$ that evaluated to $\top$ under given assignment σ while obtaining t
3. collect all *ite* conditions $c_0^t, \ldots, c_m^t$ that evaluated to $\perp$ under given assignment σ while obtaining t

Lemma

$$\bigwedge_{i=0}^{j} c_i^s \wedge \bigwedge_{i=0}^{k} \neg c_i^s \wedge \bigwedge_{i=0}^{l} c_i^t \wedge \bigwedge_{i=0}^{m} \neg c_i^t \rightarrow s = t$$

Violation of rule B

$s := \lambda_{\bar{y}}(a_1, \ldots, a_n) \in \rho(\lambda_{\bar{x}}),\ t := \lambda_{\bar{x}}[x_1 \backslash a_1, \ldots, x_n \backslash a_n]_{\mathbf{p}}$ violates rule B

1. collect conditions $c_0^s, \ldots, c_j^s,\ c_0^s, \ldots, c_k^s$ as before
2. collect all *ite* conditions $c_0^t, \ldots, c_l^t$ that evaluated to $\top$ under given assignment σ while obtaining t
3. collect all *ite* conditions $c_0^t, \ldots, c_m^t$ that evaluated to $\perp$ under given assignment σ while obtaining t

Lemma

$$\underbrace{\bigwedge_{i=0}^{j} c_i^s \ \wedge \bigwedge_{i=0}^{k} \neg c_i^s}_{\text{Prop. conditions s}} \ \wedge \bigwedge_{i=0}^{l} c_i^t \ \wedge \bigwedge_{i=0}^{m} \neg c_i^t \rightarrow s = t$$

Violation of rule B

$s := \lambda_{\bar{y}}(a_1, \ldots, a_n) \in \rho(\lambda_{\bar{x}}),\ t := \lambda_{\bar{x}}[x_1 \backslash a_1, \ldots, x_n \backslash a_n]_{\mathbf{p}}$ violates rule B

1. collect conditions $c_0^s, \ldots, c_j^s,\ c_0^s, \ldots, c_k^s$ as before
2. collect all *ite* conditions $c_0^t, \ldots, c_l^t$ that evaluated to $\top$ under given assignment σ while obtaining t
3. collect all *ite* conditions $c_0^t, \ldots, c_m^t$ that evaluated to $\perp$ under given assignment σ while obtaining t

Lemma

$$\bigwedge_{i=0}^{j} c_i^s \;\wedge\; \bigwedge_{i=0}^{k} \neg c_i^s \;\wedge\; \underbrace{\bigwedge_{i=0}^{l} c_i^t \;\wedge\; \bigwedge_{i=0}^{m} \neg c_i^t}_{\text{Eval. conditions t}} \rightarrow s = t$$

Violation of rule B

$s := \lambda_{\bar{y}}(a_1, \ldots, a_n) \in \rho(\lambda_{\bar{x}}),\ t := \lambda_{\bar{x}}[x_1 \backslash a_1, \ldots, x_n \backslash a_n]_{\mathbf{p}}$ violates rule B

1. collect conditions $c_0^s, \ldots, c_j^s,\ c_0^s, \ldots, c_k^s$ as before
2. collect all *ite* conditions $c_0^t, \ldots, c_l^t$ that evaluated to $\top$ under given assignment σ while obtaining t
3. collect all *ite* conditions $c_0^t, \ldots, c_m^t$ that evaluated to $\bot$ under given assignment σ while obtaining t

Lemma

$$\bigwedge_{i=0}^{j} c_i^s \ \wedge \bigwedge_{i=0}^{k} \neg c_i^s \ \wedge \bigwedge_{i=0}^{l} c_i^t \ \wedge \bigwedge_{i=0}^{m} \neg c_i^t \rightarrow s = t$$

abstr. variable consistency

3 benchmark categories

- crafted: benchmarks with SMT-LIB v2 macros
- SMT'12: all non-extensional QF_AUFBV benchmarks used in SMT competition 2012
- application: instantiation benchmarks [1] [Falke et al., 2013] generated with LLBMC (with and without λ-terms as arrays)

SMT Solvers

- Boolector: with DP_λ
- Boolector_{nop}: with DP_λ, but without rule P
- Boolector_β: with eager λ-term elimination
- Boolector_{sc12}: version submitted to SMT competition 2012
- CVC4 1.2, MathSAT 5.2.6, SONOLAR 2013-05-15, STP 1673 (svn revision), Z3 4.3.1

Machine Setup: 2.83Ghz Intel Core 2 Quad, 8GB memory, Ubuntu 12.04.2

[1] http://llbmc.org/files/downloads/vstte-2013.tgz

	Solver	Solved	TO	MO	Time [10^3s]	Space [GB]
macro blow-up	Boolector	100	0	0	24.2	9.4
	Boolector_{nop}	100	0	0	18.2	8.4
	Boolector_{β}	28	49	23	91.5	160.0
	CVC4	21	0	79	95.7	551.6
	MathSAT	51	2	47	64.6	395.0
	SONOLAR	26	74	0	90.2	1.7
	Z3	21	0	79	95.0	552.2
fisher-yates SAT	Boolector	7	10	1	14.0	7.5
	Boolector_{nop}	4	13	1	17.3	7.0
	Boolector_{β}	6	1	11	15.0	76.4
	CVC4	5	1	12	15.7	83.6
	MathSAT	6	10	2	14.7	17.3
	SONOLAR	3	14	1	18.1	6.9
	Z3	6	12	0	14.8	0.2
fisher-yates UNSAT	Boolector	5	13	1	17.4	7.1
	Boolector_{nop}	4	14	1	18.2	6.9
	Boolector_{β}	9	0	10	12.1	72.0
	CVC4	3	4	12	19.2	82.1
	MathSAT	6	11	2	15.9	14.7
	SONOLAR	3	15	1	19.2	6.8
	Z3	10	9	0	11.2	2.2

Limits: time: 1200s, memory: 7GB

Penalty: TO: +1200s, MO: +1200s, +7GB

	Solver	Solved	TO	MO	Time [10^3s]	Space [GB]
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	MathSAT	6	11	2	15.9	14.7
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Limits: time: 1200s, memory: 7GB
Penalty: TO: +1200s, MO: +1200s, +7GB

	Solver	Solved	TO	MO	Time [10^3s]	Space [GB]
SMT'12	Boolector	139	10	0	19.9	14.8
	Boolector$_{nop}$	134	15	0	26.3	14.5
	Boolector$_{\beta}$	137	11	1	21.5	22.7
	Boolector$_{sc12}$	140	9	0	15.9	10.3

- Boolector solves 5 instances that Boolector$_{\beta}$ couldn't
- Boolector$_{\beta}$ solves 3 instances that Boolector couldn't
- combined they solve 2 instances that Boolector$_{sc12}$ couldn't

Limits: time: 1200s, memory: 7GB
Penalty: TO: +1200s, MO: +1200s, +7GB

	Solver	Solved	TO	MO	Time [s]	Space [MB]
instantiation	Boolector	37	8	0	576	235
	Boolector_{nop}	35	10	0	673	196
	Boolector_{β}	44	1	0	138	961
	Boolector_{sc12}	39	6	0	535	308
	STP	44	1	0	141	3814
lambda[2]	Boolector	37	8	0	594	236
	Boolector_{nop}	35	10	0	709	166
	Boolector_{β}	45	0	0	52	676
	Boolector_{sc12}	-	-	-	-	-
	STP	-	-	-	-	-

Limits: time: 60s, memory: 7GB
Penalty: TO: +60s, MO: +60s, +7GB

[2] *lambda* benchmarks kindly provided by Carsten Sinz et. al.

	Solver	Solved	TO	MO	Time [s]	Space [MB]
instantiation	Boolector	37	8	0	576	235
	Boolector_{nop}	35	10	0	673	196
	Boolector_{β}	44	1	0	138	961
	Boolector_{sc12}	39	6	0	535	308
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	Boolector_{sc12}	-	-	-	-	-
	STP	-	-	-	-	-

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	STP	-	-	-	-	-

Limits: time: 60s, memory: 7GB
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Conclusion

- DP$_\lambda$, a decision procedure for non-recursive, non-extensional λ-terms
  $\rightarrow$ *consistent*$_\lambda$, *lemma*$_\lambda$
- experimental results look promising
  $\rightarrow$ category application demonstrates potential of native λ-term support
- still room for improvements
  $\rightarrow$ optimization of β-reduction
  $\rightarrow$ no λ-term specific rewriting yet

Future Work

- rewriting rules for λ-terms
- better β-reduction implementation
- various β-reduction strategies
- extensionality on λ-terms
- quantifiers

Lemmas on Demand for Lambdas

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Portland, OR, USA

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