$\forall$Exp+Res Does not P-Simulate Q-resolution

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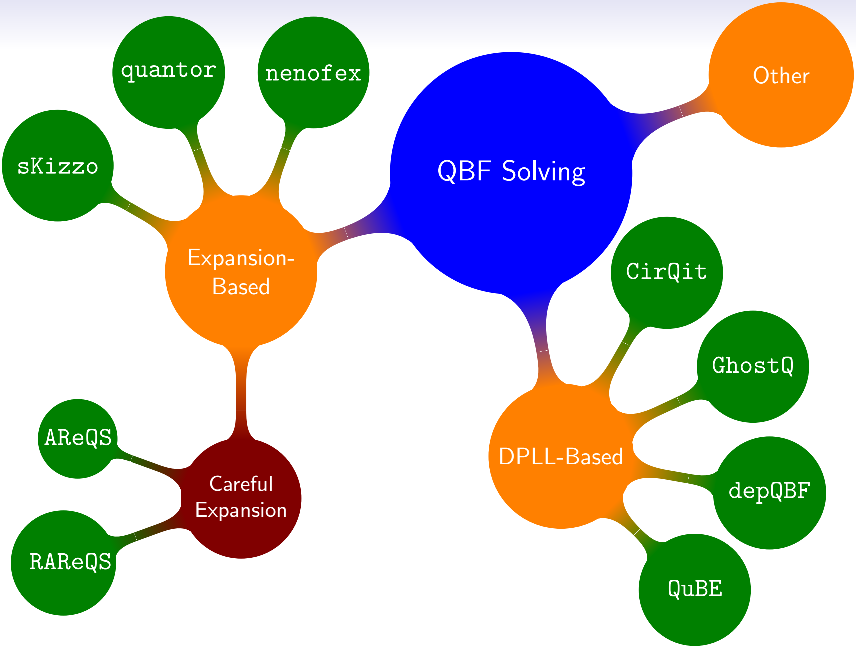
SAT 2013, July 8-12


QBF Solving
Other
Expansion-Based
quantor
nenofex
sKizzo
Careful Expansion
AReQS
RAReQS
DPLL-Based
CirQit
GhostQ
depQBF
QuBE

Quantified Boolean Formula (QBF)

- an extension of SAT with quantifiers

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- we consider prenex form with maximal blocks of variables

$$\forall \mathcal{U}_1 \exists \mathcal{E}_2 \ldots \forall \mathcal{U}_{2N-1} \exists \mathcal{E}_{2N}.\ \phi$$

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Solving

- DPLL — Q-Resolution (`QuBE`, `depqbf`, etc.)
- Expansion — ?? (`Quantor`, `sKizzo`, `Nenofex`)
  - "Careful" expansion (`AReQS`,`RAReQS`)

Q-resolution

Q-resolution = *Q-resolution rule* + ∀*-reduction*

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Q-resolution rule

C_1, C_2 with one and only one complementary literal l, where l is existential

- derive $C_1 \cup C_2 \smallsetminus \{l, \bar{l}\}$

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$\forall$-reduction

- if $k \in C$ is universal with highest level in C, remove k from C

Expansion

$$\forall x.\ \Phi = \Phi[x/0] \wedge \Phi[x/1]$$

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Fresh variables in order to keep prenex form

$$\exists e_1 \forall u_2 \exists e_3.\ (\bar{e}_1 \vee e_3) \wedge (\bar{e}_3 \vee e_1) \wedge (u_2 \vee e_3) \wedge (\bar{u}_2 \vee \bar{e}_3)$$

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$$\begin{aligned} \exists e_1 e_3^{u_2/0} e_3^{u_2/1}.\quad & (\bar{e}_1 \vee e_3^{u_2/0}) \wedge (\bar{e}_3^{u_2/0} \vee e_1) \wedge \\ & (\bar{e}_1 \vee e_3^{u_2/1}) \wedge (\bar{e}_3^{u_2/1} \vee e_1) \wedge \\ & e_3^{u_2/0} \wedge \\ & \bar{e}_3^{u_2/1} \end{aligned}$$

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1. Expand all universal variables
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Only certain values may be needed:

$$\forall u \exists e.\, (u \vee e) \wedge (u \vee \bar{e}) \wedge (\bar{u} \vee e) \quad \text{(false)}$$

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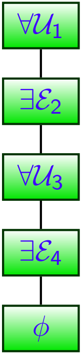
$$\exists e^{u/0} e^{u/1}.\ e^{u/0} \wedge \bar{e}^{u/0} \wedge e^{u/1} \quad \text{(false full)}$$
$$\exists e^{u/0}.\ e^{u/0} \wedge \bar{e}^{u/0} \quad \text{(false partial)}$$

Recursive Partial Expansion



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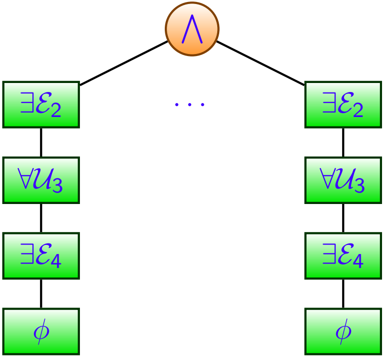
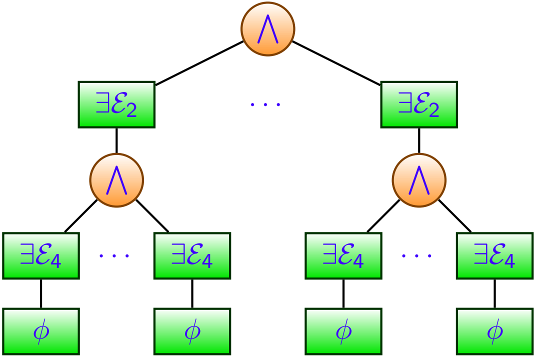


Recursive Partial Expansion



$\forall$Exp+Res

Proof: $(\mathcal{T}, \pi)$

∀Exp+Res

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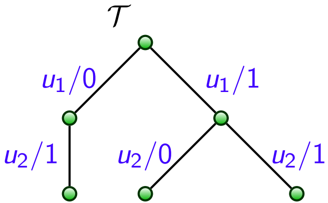
(1) Expansion tree $\mathcal{T}$: for each block of variables it tells us how to expand it.



∀Exp+Res

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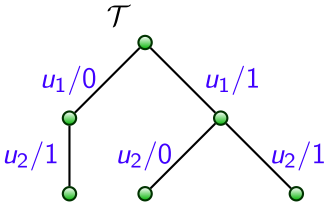

(2) Propositional Resolution Refutation π of expansion resulting from the expansion tree $\mathcal{T}$.

Performing Expansion

- For a clause $C = e_i \vee u \vee e_k$, for $\tau = \tau_1, \ldots, \tau_n$

$$\begin{array}{lcll} \mathscr{E}(\tau_1, \ldots, \tau_n,\ C) & = & e_i^{\tau_1, \ldots, \tau_{i/2}} \vee e_k^{\tau_1, \ldots, \tau_{k/2}} & \text{if } u[\tau] = 0 \\ \mathscr{E}(\tau_1, \ldots, \tau_n,\ C) & = & 1 & \text{if } u[\tau] = 1 \end{array}$$

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- For an expansion tree $\mathcal{T}$ and a matrix ϕ consider the union of clauses $\mathscr{E}(\tau, C)$ for all branches $\tau \in \mathcal{T}$ and $C \in \phi$.

Separation Formula

$n = 1$

$$\exists e_1 \forall u_1 \exists c_1 c_2 .$$
$$(\bar{e}_1 \vee c_1) \wedge (\bar{u}_1 \vee c_1) \wedge$$
$$(e_1 \vee c_2) \wedge (u_1 \vee c_2) \wedge$$
$$\bar{c}_1 \vee \bar{c}_2$$

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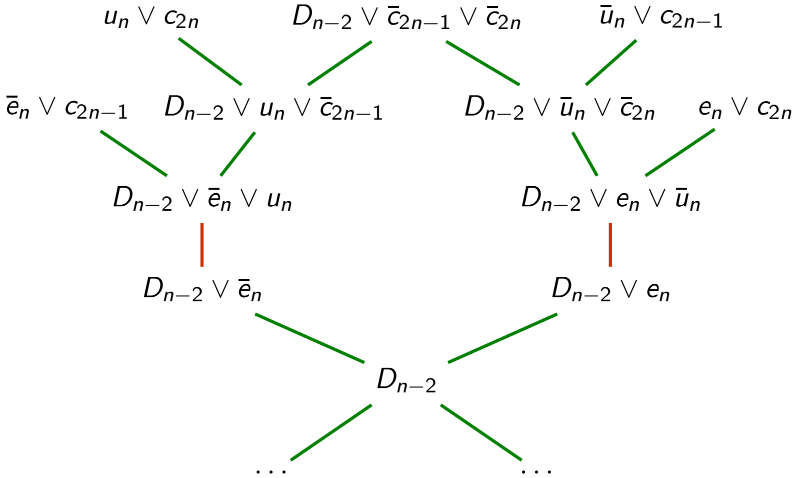
$$\bar{c}_1 \vee \bar{c}_2$$

General case

$$\exists e_1 \forall u_1 \exists c_1 c_2 \ldots \exists e_n \forall u_n \exists c_{2n-1} c_{2n} .$$
$$\bigwedge_{i \in 1..n} (\bar{e}_i \vee c_{2i-1}) \wedge (\bar{u}_i \vee c_{2i-1}) \wedge$$
$$\bigwedge_{i \in 1..n} (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i}) \wedge$$
$$\bigvee_{i \in 1..2n} \bar{c}_i$$

Q-Resolution Proof



Expansion

$$\exists e_2 \forall u_2 \exists c_3 c_4 \ldots \exists e_N \forall u_N \exists c_{2n-1} c_{2n}.$$
$$(\bar{e}_1 \vee c_1^{u_1/1}) \wedge (c_1^{u_1/1}) \wedge (e_1 \vee c_2^{u_1/1}) \wedge$$
$$\textstyle\bigwedge_{i \in 2..N} (\bar{e}_i \vee c_{2i-1}) \wedge (\bar{u}_i \vee c_{2i-1}) \wedge \tag{1}$$
$$\textstyle\bigwedge_{i \in 2..N} (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i}) \wedge$$
$$\bar{c}_1^{u_1/1} \vee \bar{c}_2^{u_1/1} \vee \textstyle\bigvee_{i \in 3..2n} \bar{c}_i$$

$$\exists e_2 \forall u_2 \exists c_3 c_4 \ldots \exists e_N \forall u_N \exists c_{2n-1} c_{2n}.$$
$$(\bar{e}_1 \vee c_1^{u_1/0}) \wedge (c_2^{u_1/0}) \wedge (e_1 \vee c_2^{u_1/0}) \wedge$$
$$\textstyle\bigwedge_{i \in 2..N} (\bar{e}_i \vee c_{2i-1}) \wedge (\bar{u}_i \vee c_{2i-1}) \wedge \tag{2}$$
$$\textstyle\bigwedge_{i \in 2..N} (e_i \vee c_{2i}) \wedge (u_i \vee c_{2i}) \wedge$$
$$\bar{c}_1^{u_1/0} \vee \bar{c}_2^{u_1/0} \vee \textstyle\bigvee_{i \in 3..2n} \bar{c}_i$$

- (1) $\Rightarrow e_1$ and (2) $\Rightarrow \neg e_1$.

The formula (1) $\wedge\ \bar{e}_1$:

$$(c_1^{u_1/1}) \wedge (c_2^{u_1/1}) \wedge$$
$$\exists e_2 \forall u_2 \exists c_3 c_4 \ldots \exists e_N \forall u_N \exists c_{2N-1} c_{2N}.$$
$$\bigwedge_{i \in 2..N} (\bar{e}_i \vee c_{2N-1}) \wedge (\bar{u}_i \vee c_{2N-1}) \wedge$$
$$\bigwedge_{i \in 2..N} (e_i \vee c_{2N}) \wedge (u_i \vee c_{2N}) \wedge$$
$$\bigvee_{i \in 3..2N} \bar{c}_i$$

- The rest must be *false*.

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- $\forall$Exp+Res is an expansion-based proof system for QBF
- Conjecture: $\forall$Exp+Res and Q-resolution are incomparable [Janota and Marques-Silva, 2013].
- We have shown that $\forall$Exp+Res does not simulate Q-resolution.

Thank you for your attention!

Questions?

Janota, M. and Marques-Silva, J. (2013).
On propositional QBF expansions and Q-resolution.
In *SAT '13*.