Reasoning Engines for Rigorous System Engineering

Block 3: Quantified Boolean Formulas and DepQBF

1. Inside Search-based QBF Solvers

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Overview (1/2)

Success Story of SAT Solving:

- Backtracking search, clause learning as systematic application of resolution: CDCL
- Broad field of research: solver technology, theory, applications.
- Solver development driven by applications and vice versa.

Quantified Boolean Formulae (QBF):

- Explicit quantifiers $(\forall, \exists)$ over propositional variables.
- NP-completeness of SAT vs. PSPACE-completeness of QBF.
- Potentially more succinct QBF encodings of PSPACE-complete problems.

QBF Solving:

- QCDCL: inspired by CDCL for SAT.
- Alternative: variable elimination.
- After peak in 2006/2007, renewed interest in QBF.

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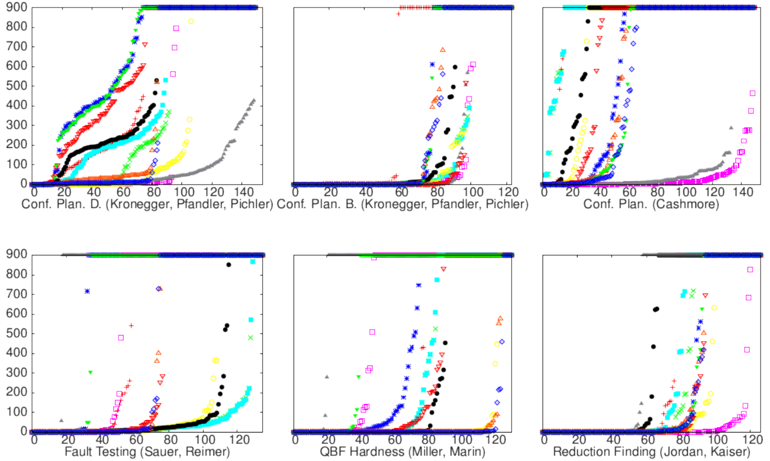
Solver Performance in the QBF Gallery 2013



- 6 new formula sets, 150 formulas each.
- At least one solver is good for a set (but it is not always the same).
- `http://www.kr.tuwien.ac.at/events/qbfgallery2013/`

Overview (2/2)

Our Focus:

- Search-based QBF solving as a major approach (next to variable elimination).
- Bottom-up approach: from basic building blocks to general view.
- The role of Q-resolution in QBF solvers.

Lessons to be Learned:

- (Q)CDCL is not just a combination of backtracking search and clause learning.
- Implementation: QBF solvers are more complex than SAT solvers.
- Pitfalls when porting SAT solver technology to QBF.

Example: DepQBF

- Search-based QBF solver, under active development since 2010.
- Open source: `http://lonsing.github.io/depqbf/`
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Syntax, Semantics, Notation

QBF Syntax

QBF in Prenex Conjunctive Normal Form:

- Given a Boolean formula $\phi(x_1, \ldots, x_m)$ in CNF.
- Quantifier prefix $Q_1 B_1 Q_2 B_2 \ldots Q_m B_m$.
- Quantifiers $Q_i \in \{\forall, \exists\}$.
- Quantifier block $B_i \subseteq \{x_1, \ldots, x_m\}$ containing variables.
- QBF in prenex CNF (PCNF): $Q_1 B_1 Q_2 B_2 \ldots Q_m B_m.\phi(x_1, \ldots, x_m)$.
- $B_i \leq B_{i+1}$: quantifier blocks are linearly ordered (extended to variables, literals).

Example

- Given the CNF $\phi := (x_1 \vee \neg x_3) \wedge (x_1 \vee x_4) \wedge (\neg x_2 \vee \neg x_3) \wedge (\neg x_3 \vee x_4)$.
- Given the quantifier prefix $\forall x_1, x_2 \exists x_3, x_4$.
- Prenex CNF: $\forall x_1, x_2 \exists x_3, x_4.(x_1 \vee \neg x_3) \wedge (x_1 \vee x_4) \wedge (\neg x_2 \vee \neg x_3) \wedge (\neg x_3 \vee x_4)$.

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QBF Semantics (1/5)

Variable Assignments:

- Mapping $V \to \{\top, \bot\}$, variables $V = \{x_1, \ldots, x_m\}$, truth values $\top$ and $\bot$.
- Given a CNF $\phi(x_1, \ldots, x_i, \ldots, x_m)$, assigning x_i to v, where $v \in \{\top, \bot\}$, produces the CNF $\phi(x_1, \ldots, x_m)[x_i/v]$.
- In $\phi(x_1, \ldots, x_m)[x_i/v]$, occurrences of x_i are replaced by the value v.
- Standard simplifications by Boolean algebra: $\phi' \vee \top \equiv \top$, $\phi' \vee \bot \equiv \phi'$, $\phi' \wedge \top \equiv \phi'$, $\phi' \wedge \bot \equiv \bot$.
- Write $\phi[x_i]$ for $\phi[x_i/\top]$ and $\phi[\neg x_i]$ for $\phi[x_i/\bot]$: literals denote assignments.

Example

- Given the CNF $\phi := (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$.
- $\phi[x_2/\top] = (x_1 \vee \neg\top) \wedge (\top \vee x_3) \wedge (\neg x_1 \vee \neg x_3)$.
- $\phi[x_2/\top] = (x_1) \wedge (\neg x_1 \vee \neg x_3)$.

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QBF Semantics (2/5)

Recursive Definition:

- Recursively assign the variables in prefix order (from left to right).
- Base cases: the QBF $\top$ ($\bot$) is satisfiable (unsatisfiable).
- $\psi = \forall x \ldots \phi$ is satisfiable if $\psi[\neg x]$ and $\psi[x]$ are satisfiable.
- $\psi = \exists x \ldots \phi$ is satisfiable if $\psi[\neg x]$ or $\psi[x]$ is satisfiable.
- Prerequisite: every variable is quantified in the prefix (no free variables).
- Satisfiability-equivalence of two PCNFs ψ and ψ': $\psi \equiv \psi'$.

Example

The PCNF $\psi = \forall x \exists y.(x \vee \neg y) \wedge (\neg x \vee y)$ is satisfiable if

(1) $\psi[x] = \exists y.(y)$ and

(2) $\psi[\neg x] = \exists y.(\neg y)$ are satisfiable.

(1) $\psi[x] = \exists y.(y)$ is satisfiable since $\psi[x, y] = \top$ is satisfiable.
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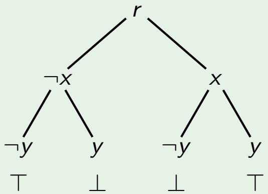
QBF Semantics (4/5)

Assignment Trees of a PCNF ψ:

- Dedicated root node r.
- Each path from root r to a leaf represents a variable assignment A.
- The assignment sequence along each path follows the prefix order.
- Leaf is labelled with $\top$ if the PCNF $\psi[A]$ is satisfiable.
- Leaf is labelled with $\bot$ if the PCNF $\psi[A]$ is unsatisfiable.

Example

- Satisfiable PCNF $\psi = \forall x \exists y.(x \lor \neg y) \land (\neg x \lor y)$.
- The node "r" represents ψ.
- The node "$\neg x$" represents $\psi[\neg x] = \exists y.(\neg y)$.
- Leftmost path: $\psi[\neg x, \neg y]$ is satisfiable.
- Rightmost path: $\psi[x, y]$ is satisfiable.



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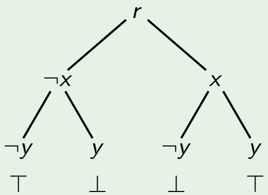

QBF Semantics (5/5)

Example (continued)

- Satisfiable PCNF $\psi = \forall x \exists y.(x \vee \neg y) \wedge (\neg x \vee y)$.
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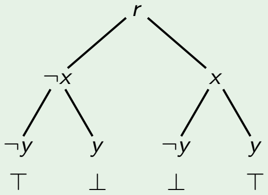

- Assignment trees visualize the structure of recursive semantical evaluation.
- $\forall x.\psi$: both recursive subcases $\psi[\neg x]$ and $\psi[x]$ (children) must be satisfiable.
- $\exists x.\psi$: one recursive subcase $\psi[\neg x]$ or $\psi[x]$ (child) must be satisfiable.

Basic Backtracking Search

Recursive QBF Semantics and Backtracking Search

- Application of semantic rules: "splitting", "decision making", "branching".
- Backtracking: flip value of decision (wrt. quantifier type and base case).
- Early termination if one subcase of ∃ (∀) is satisfiable (unsatisfiable).

```
bool bt_search (PCNF Qxψ, Assignment A)
   /* 1. Simplify under given assignment. */
       ψ' := simplify(Qxψ[A]);
   /* 2. Check base cases. */
       if (ψ' == ⊥)
         return false;
       if (ψ' == ⊤)
         return true;
   /* 3. Decision making, backtracking. */
       if (Q == ∃)
         return bt_search (ψ', A ∪ {¬x}) ||
                bt_search (ψ', A ∪ {x});
       if (Q == ∀)
         return bt_search (ψ', A ∪ {¬x}) &&
                bt_search (ψ', A ∪ {x});
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M. Davis, G. Logemann, D. Loveland. A Machine Program for Theorem-Proving. Commun. ACM, 1962.

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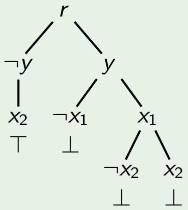
Backtracking Search: Example

Example

$\psi := \forall y \exists x_1, x_2.(x_2) \wedge (\neg y \vee \neg x_2) \wedge (\neg y \vee x_1)$.

- Leftmost path: check $\forall$-subcase $\psi[\neg y]$.
  $\exists$-subcase $\psi[\neg y, x_2] = \top$ already, no need to try $\psi[\neg y, \neg x_2]$.
  Backtrack and check $\psi[y]$.
- $\exists$-subcase $\psi[y, \neg x_1] = \bot$, flip x_1 and check $\psi[y, x_1]$.
- Both $\exists$-subcases $\psi[y, x_1, \neg x_2] = \bot$ and $\psi[y, x_1, x_2] = \bot$, hence $\exists$-subcase $\psi[y, x_1]$ unsat.
- Since $\exists$-subcases $\psi[y, \neg x_1]$, $\psi[y, x_1]$ unsat., also $\forall$-subcase $\psi[y]$ and ψ unsat.



Observation: the clause (x_2) in ψ can only be satisfied by setting x_2 to true.

- Every subcase $\psi[\ldots, \neg x_2, \ldots]$ is unsatisfiable: consider $\psi[\ldots, x_2, \ldots]$ instead.
- $\forall$-subcase $\psi[x_2, y] = \bot$, hence ψ unsatisfiable: smaller assignment tree!

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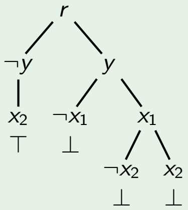

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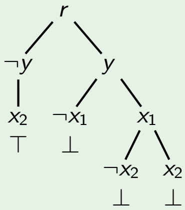

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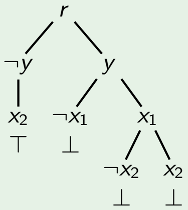

Observation: the clause (x_2) in ψ can only be satisfied by setting x_2 to true.

- Every subcase $\psi[\ldots, \neg x_2, \ldots]$ is unsatisfiable: consider $\psi[\ldots, x_2, \ldots]$ instead.
- $\forall$-subcase $\psi[x_2, y] = \bot$, hence ψ unsatisfiable: smaller assignment tree!

Backtracking Search: Example

Example

$\psi := \forall y \exists x_1, x_2.(x_2) \wedge (\neg y \vee \neg x_2) \wedge (\neg y \vee x_1)$.

- Leftmost path: check $\forall$-subcase $\psi[\neg y]$.
  $\exists$-subcase $\psi[\neg y, x_2] = \top$ already, no need to try $\psi[\neg y, \neg x_2]$.
  Backtrack and check $\psi[y]$.
- $\exists$-subcase $\psi[y, \neg x_1] = \bot$, flip x_1 and check $\psi[y, x_1]$.
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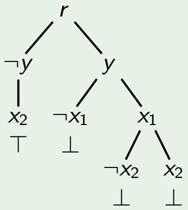

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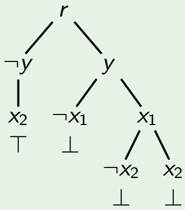

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- Assignments by decisions only (i.e. must be flipped during backtracking).
- Generated assignment trees might contain irrelevant branches.
- Goal: add rules to make assignments other than decisions which can be ignored during backtracking.
- Avoid irrelevant branches resulting from "wrong" decisions.

Example (continued)

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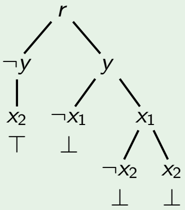

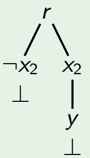

Observe: $\exists x_2$ *is not leftmost in prefix of* ψ.

Improvements to Backtracking Search:
Assignment Generation

Towards Modern Search-Based QBF Solving (1/5)

Definition (Unit Literal Detection)

- Given a QBF ψ, a clause $C \in \psi$ is *unit* if and only if $C = (l)$ and $q(l) = \exists$.
- The existential literal l in C is called a *unit literal*.
- *Unit literal detection* $UL(C) := \{l\}$ collects the assignment $\{l\}$ from the unit clause $C = (l)$.
- Unit literal detection on a QBF ψ: $UL(\psi) := \bigcup_{C \in \psi} UL(C)$.

Example (continued)

The clause (x_2) in $\psi := \forall y \exists x_1, x_2.(x_2) \wedge (\neg y \vee \neg x_2) \wedge (\neg y \vee x_1)$ is unit: $UL(\psi) = \{x_2\}$

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Towards Modern Search-Based QBF Solving (2/5)

Definition (Pure Literal Detection)

- A literal l is *pure* in a QBF ψ if there are clauses which contain l but no clauses which contain $\neg l$.
- *Pure literal detection* $PL(\psi) := \bigcup\{l'\}$ collects the assignment $\{l'\}$ such that l is a pure literal in ψ and $l' := l$ if $q(l) = \exists$ and $l' := \neg l$ if $q(l) = \forall$.
- The variable of an existential (universal) pure literal is assigned so that clauses are satisfied (not satisfied) by that assignment.

Example (continued)

The universal literal $\neg y$ in $\psi := \forall y \exists x_1, x_2.(x_2) \wedge (\neg y \vee \neg x_2) \wedge (\neg y \vee x_1)$ is pure. $PL(\psi) = \{y\}$ and $\psi[y] := \exists x_1, x_2.(x_2) \wedge (\neg x_2) \wedge (x_1)$.

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Towards Modern Search-Based QBF Solving (3/5)

Definition

Given a clause C, *universal reduction (UR)* on C produces the clause

$$UR(C) := C \setminus \{l \in C \mid q(l) = \forall \text{ and } \forall l' \in C \text{ with } q(l') = \exists : var(l') < var(l)\},$$

where $<$ is the linear variable ordering given by the quantifier prefix.

- UR deletes "trailing" universal literals from clauses.
- UR shortens clauses.

Example (continued)

Given $\psi := \forall y \exists x_1, x_2.(x_2) \wedge (\neg y \vee \neg x_2) \wedge (\neg y \vee x_1)$.
By UL: $\psi[x_2] := \forall y \exists x_1.(\neg y) \wedge (\neg y \vee x_1)$.
$UR((\neg y)) = \emptyset$ in $\psi[x_2]$.

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Definition

Boolean Constraint Propagation for QBF (QBCP):

- Given a PCNF ψ and the empty assignment $A = \{\}$, i.e. $\psi[A] = \psi$.
  1. Apply universal reduction to $\psi[A]$.
  2. Apply unit literal detection (UL) to $\psi[A]$ to get new assignments by UL.
  3. Apply pure literal detection (PL) to $\psi[A]$ to find new assignments by PL.
- Add assignments found by UL and PL to A, repeat steps 1-3.
- Stop if A does not change anymore or if $\psi[A] = \top$ or $\psi[A] = \bot$.

Towards Modern Search-Based QBF Solving (5/5)

Properties of QBCP:

- QBCP takes a PCNF ψ and an assignment A and produces an extended assignment A' and a new PCNF $\psi' = \psi[A']$ by UL, PL, and UR.
- Soundness: $\psi \equiv \psi'$ (satisfiability-equivalence).
- No order restriction: QBCP assigns variables from any quantifier block.

QBCP in Practice:

- Combine decision making and QBCP.
- Successively apply QBCP starting with $A = \{x\}$ where x is a decision.
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QBCP Example

Example

- $\psi =$ $\forall y_5 \exists x_1 \forall y_2 \exists x_3, x_4. (\neg y_5 \lor x_4) \land (y_5 \lor \neg x_4) \land (x_1 \lor y_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg y_2 \lor \neg x_3)$.
- No simplifications of ψ by QBCP possible.
- Make decision: $A = \{y_5\}$.
- $\psi[y_5] = \exists x_1 \forall y_2 \exists x_3, x_4. (x_4) \land (x_1 \lor y_2 \lor \neg x_4) \land (\neg x_1 \lor x_3 \lor \neg x_4) \land (\neg y_2 \lor \neg x_3)$.
- By UL: $\psi[y_5, x_4] = \exists x_1 \forall y_2 \exists x_3. (x_1 \lor y_2) \land (\neg x_1 \lor x_3) \land (\neg y_2 \lor \neg x_3)$.
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- By PL: $\psi[y_5, x_4, y_2] = \exists x_1 \exists x_3. (x_1) \land (\neg x_1 \lor x_3) \land (\neg x_3)$.
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- By QBCP, we have shown: $\psi[y_5] \equiv \psi[y_5, x_4, y_2, x_1, x_3] \equiv \bot$.
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- No simplifications of ψ by QBCP possible.
- Make decision: $A = \{y_5\}$.
- $\psi[y_5] = \exists x_1 \forall y_2 \exists x_3, x_4 . (x_4) \wedge (x_1 \vee y_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (\neg y_2 \vee \neg x_3)$.
- By UL: $\psi[y_5, x_4] = \exists x_1 \forall y_2 \exists x_3 . (x_1 \vee y_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg y_2 \vee \neg x_3)$.
- By UR: $\psi[y_5, x_4] = \exists x_1 \forall y_2 \exists x_3 . (x_1) \wedge (\neg x_1 \vee x_3) \wedge (\neg y_2 \vee \neg x_3)$.
- By PL: $\psi[y_5, x_4, y_2] = \exists x_1 \exists x_3 . (x_1) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_3)$.
- By UL: $\psi[y_5, x_4, y_2, x_1] = \exists x_3 . (x_3) \wedge (\neg x_3)$.
- By UL: $\psi[y_5, x_4, y_2, x_1, x_3] = \perp$.
- By QBCP, we have shown: $\psi[y_5] \equiv \psi[y_5, x_4, y_2, x_1, x_3] \equiv \perp$.
- Since y_5 is a universal decision: $\psi[y_5] \equiv \perp \equiv \psi$, only one branch explored.
- Worst case: search tree has 2^5 branches.

QBCP Example

Example

- $\psi =$ $\forall y_5 \exists x_1 \forall y_2 \exists x_3, x_4 . (\neg y_5 \vee x_4) \wedge (y_5 \vee \neg x_4) \wedge (x_1 \vee y_2 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4) \wedge (\neg y_2 \vee \neg x_3)$.
- No simplifications of ψ by QBCP possible.
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- By UL: $\psi[y_5, x_4, y_2, x_1, x_3] = \bot$.
- By QBCP, we have shown: $\psi[y_5] \equiv \psi[y_5, x_4, y_2, x_1, x_3] \equiv \bot$.
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- Worst case: search tree has 2^5 branches.

Iterative Search-Based QBF Solving (QDPLL)

QDPLL:

- QBF-specific variant of the DPLL algorithm for propositional logic [DLL62].
- Original descriptions [GNT01, CGS98] both recursive and iterative.
- Start with empty assignment.
- Decisions open a new $\exists/\forall$-subcase.
- Function qbcp applies UL, PL, UR and simplifications to extend the assignment corresponding to the current $\exists/\forall$-subcase.
- Function `analyze`: retraction of assignments, flipping a decision variable by backtracking.

```
Result qdpll (PCNF f)
  Result r = UNDEF;
  Assignment a = {};
  while (true)
    /* Simplify. */
    (r,a) = qbcp (f,a);
    if (r == UNDET)
      /* Decision making. */
      a = assign_dec_var (f,a);
    else
      /* Backtracking. */
      /* r == UNSAT or r == SAT */
      btlevel = analyze (r,a);
      if (btlevel == INVALID)
        return r;
      else
        a = backtrack (btlevel);
```

M. Cadoli, A. Giovanardi, M. Schaerf. An Algorithm to Evaluate Quantified Boolean Formulae. AAAI, 1998.

E. Giunchiglia, M. Narizzano, A. Tacchella. QUBE: A System for Deciding Quantified Boolean Formulas Satisfiability. IJCAR, 2001.

Search-Based QBF Solving: Iterative vs. Recursive

```
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```

```
bool bt_search (PCNF Qxψ, Assignment A)
   /* 1. Simplify under given assignment. */
       ψ' := simplify(Qxψ[A]);
   /* 2. Check base cases. */
       if (ψ' == ⊥)
         return false;
       if (ψ' == ⊤)
         return true;
   /* 3. Decision making, backtracking. */
       if (Q == ∃)
         return bt_search (ψ', A ∪ {¬x}) ||
                bt_search (ψ', A ∪ {x});
       if (Q == ∀)
         return bt_search (ψ', A ∪ {¬x}) &&
                bt_search (ψ', A ∪ {x});
```

Comparison:

- bt_search very close to recursive semantics.
- qdpll explicitly enumerates paths (i.e. assignments) in assignment trees.
- QBCP makes the difference between qdpll and bt_search.
- Structure of qdpll is close to implementations of modern QBF solvers.

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- Structure of qdpll is close to implementations of modern QBF solvers.

Improvements to Backtracking Search:
Backtracking is not optimal

A Closer Look on Backtracking (1/2)

Assignments:

- Represented as sequence $A = \{l_1, l_2, \ldots, l_n\}$ of literals.
- Assignments due to decisions and QBCP (UL, PL).
- Literals l_i are ordered chronologically as they were assigned.
- Conflict: assignment A such that $\psi[A] = \bot$.
- Solution: assignment A such that $\psi[A] = \top$.

A Closer Look on Backtracking (2/2)

Chronological Backtracking:

- Given a conflict $A = \{\dots, d, \dots, l_n\}$ where d is the most-recent *unflipped* existential decision.
- Given a solution $A = \{\dots, d, \dots, l_n\}$ where d is the most-recent *unflipped* universal decision.
- No such d: formula solved.
- Retract decision d and later assignments: $A' = A \setminus \{d, \dots, l_n\}$.
- Set the variable of d to the opposite value (flip): $A' = A' \cup \{\neg d\}$.
- Continue with $A = A'$.

Snippet of `qdpll`:

```
/* Backtracking. */
/* r == UNSAT or r == SAT */
   btlevel = analyze (r,a);
   if (btlevel == INVALID)
     return r;
   else
     a = backtrack (btlevel);
```

QDPLL with Chronological Backtracking (1/2)

Example

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6.(\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

1. Assume that ϕ contains further clauses.
2. Decision on x_1: $A = A \cup \{x_1\}$.
3. Decision on x_2: $A = A \cup \{x_2\}$.
4. Decision on x_3: $A = A \cup \{x_3\}$.
5. $\psi[x_1, x_2, x_3] = \exists x_4 \forall y_5 \exists x_6.(x_4) \wedge (\neg x_4 \vee x_6) \wedge (y_5 \vee \neg x_6)$.
6. By QBCP (UL): $A = A \cup \{x_4, x_6\}$.
7. By QBCP (UR): conflict $A = \{x_1, x_2, x_3, x_4, x_6\}$, $\psi[A] = \bot$.
8. Flip x_3, get conflict $A = \{x_1, x_2, \neg x_3, x_4, x_6\}$, where again x_4, x_6 by UL.
9. Flip x_2, assume that no conflict/solution is found with $A = \{x_1, \neg x_2\}$.
10. Continue with a decision on x_3: $A = \{x_1, \neg x_2, x_3\}$ or $A = \{x_1, \neg x_2, \neg x_3\}$.
11. In any case, get a conflict by $\{x_1, \neg x_2, x_3, x_4, x_6\}$ and $\{x_1, \neg x_2, \neg x_3, x_4, x_6\}$.
12. Repeated subassignments $\{x_3, x_4, x_6\}$, $\{\neg x_3, x_4, x_6\}$ of conflicts (steps 7,8).
13. Flipping x_2 did not resolve the conflict, redundant work in steps 9-11.

QDPLL with Chronological Backtracking (1/2)

Example

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

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QDPLL with Chronological Backtracking (1/2)

Example

$\psi = \exists x_1,x_2,x_3,x_4 \forall y_5 \exists x_6.(\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi$.

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QDPLL with Chronological Backtracking (1/2)

Example

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \lor x_4) \land (x_3 \lor x_4) \land (\neg x_4 \lor x_6) \land (\neg x_1 \lor y_5 \lor \neg x_6) \land \phi$.

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9. Flip x_2, assume that no conflict/solution is found with $A = \{x_1, \neg x_2\}$.
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12. Repeated subassignments $\{x_3, x_4, x_6\}$, $\{\neg x_3, x_4, x_6\}$ of conflicts (steps 7,8).
13. Flipping x_2 did not resolve the conflict, redundant work in steps 9-11.

QDPLL with Chronological Backtracking (1/2)

Example

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi$.

1. Assume that ϕ contains further clauses.
2. Decision on x_1: $A = A \cup \{x_1\}$.
3. Decision on x_2: $A = A \cup \{x_2\}$.
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QDPLL with Chronological Backtracking (2/2)

Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

Conflicts generated by QDPLL:

- $A = \{x_1, x_2, x_3, x_4, x_6\}$.
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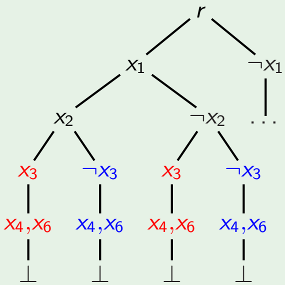
- $A = \{x_1, \neg x_2, \neg x_3, x_4, x_6\}$.
- Same conflicting subtrees after flipping x_2.
- Decision x_2 is irrelevant in this context.



Drawback of Chronological Backtracking:

- Flipping variables which are irrelevant for the current conflict/solution.
- Repeating subassignments of previous conflicts: redundant work, needless branches.

Example (continued)

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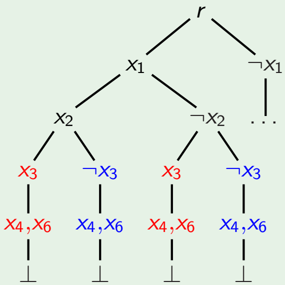

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QBCP and Implication Graphs

Definition (implication graph as a levelized graph)

- Vertices: literals in A (variable assignments), special vertex $\emptyset$ denoting a clause $C \in \psi$ such that $C[A] = \bot$ (*conflicting clause*).
- For assignments $\{l\}$ by UL from a unit clause $C[A]$: the clause $ante(l) := C$ is the *antecedent clause* of the assignment $\{l\}$.
- Define $ante(\emptyset) = C$, for a clause $C \in \psi$ such that $C[A] = \bot$.
- Edges: $(x, y) \in E$ if y assigned by UL and literal $\neg x \in ante(y)$.

Example (continued)

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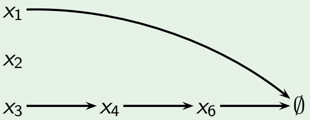
Implication graph for conflict
$A = \{x_1, x_2, x_3, x_4, x_6\}$
where x_1,x_2, and x_3 are decisions.
Note: UR applied to get $\emptyset$.



- Implication graph is implicitly constructed during QBCP.
- Similar to BCP in SAT solvers, but QBCP includes additional rules PL, UR.

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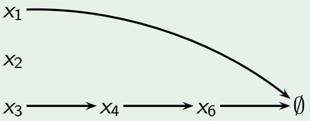

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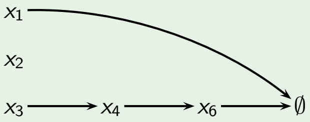
Idea: given a conflict A and the implication graph.

1. Start at the conflicting clause $\emptyset$ and traverse the implication graph backwards.
2. Collect all decisions reachable from $\emptyset$: *conflict set*.
3. Retract all assignments made after the *second* most recent existential decision in the conflict set.
4. Flip the *most* recent unflipped existential decision in the conflict set.

Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

- Conflict $A = \{x_1, x_2, x_3, x_4, x_6\}$, decisions x_1, x_2, x_3.
- Steps 1,2: conflict set $\{x_1, x_3\}$.
- Step 3: retract $\{x_2, x_3, x_4, x_6\}$ from A.
- Step 4: flip x_3, $A \cup \{\neg x_3\} = \{x_1, \neg x_3\}$.
- "Jump over" irrelevant, non-reachable x_2.



Non-Chronological Backtracking — Backjumping (1/3)

Idea: given a conflict A and the implication graph.

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Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6.(\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

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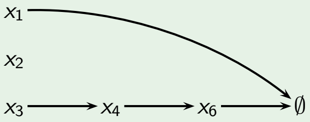
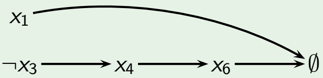


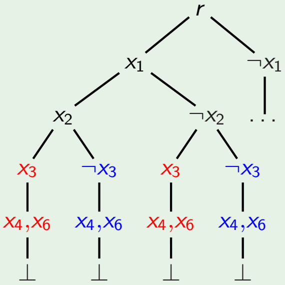
Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

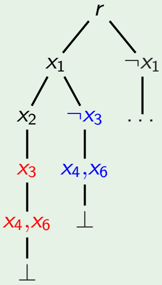
- After flipping x_3, get conflict $A = \{x_1, \neg x_3, x_4, x_6\}$, decisions $x_1, \neg x_3$.
- Conflict set $\{x_1, \neg x_3\}$.
- Retract $\{\neg x_3, x_4, x_6\}$ from A.
- Flip x_1, $A \cup \{\neg x_1\} = \{\neg x_1\}$.



Chronological backtracking:



Non-chronological backtracking:

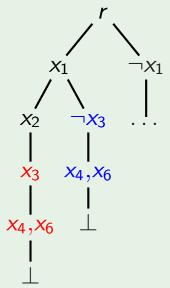


Non-Chronological Backtracking — Backjumping (3/3)

Properties of Backjumping:

- "backtracking" = "chronological backtracking".
- "backjumping" = "non-chronological backtracking".
- Potential retraction of irrelevant decisions, exponential reduction of branches in assignment trees.
- Children of nodes in assignment trees might have different labels: x_2,$\neg x_3$ in the example.
- Similar approaches to backjump from solutions, i.e. $\psi[A] = \top$.
- Fundamentally different from traditional recursive backtracking search.

Example (continued)



Snippet of `bt_search`:

```
/* 3. Decision making, backtracking. */
    if (Q == ∃)
      return bt_search (ψ′, A ∪ {¬x}) ||
             bt_search (ψ′, A ∪ {x});
    if (Q == ∀)
      return bt_search (ψ′, A ∪ {¬x}) &&
             bt_search (ψ′, A ∪ {x});
```

Implementation:

- Function analyze must be adapted.
- Stop if there is no decision to be flipped in the conflict set.
- Think of backtracking like a variant of backjumping where the conflict set always contains all decisions made.

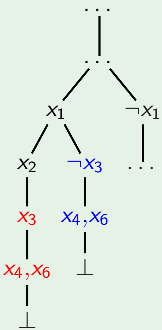
```
Result qdpll (PCNF f)
  Result r = UNDEF;
  Assignment a = {};
  while (true)
    /* Simplify. */
    (r,a) = qbcp (f,a);
    if (r == UNDET)
      /* Decision making. */
      a = assign_dec_var (f,a);
    else
      /* Backtracking. */
      /* r == UNSAT or r == SAT */
      btlevel = analyze (r,a);
      if (btlevel == INVALID)
        return r;
      else
        a = backtrack (btlevel);
```

Drawback of Backjumping

Example (continued)

$\psi = \exists x_1,x_2,x_3,x_4 \forall y_5 \exists x_6.(\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

- Assume that the assignment tree on the right is a subtree of a bigger tree.
- Observe: every assignment A with $\{x_1, x_4\} \subseteq A$ is a conflict (under QBCP).
- QBCP extends $\{x_1, x_4\}$ to $\{x_1, x_4, x_6\}$ by UL.
- Clause $(\neg x_1 \vee y_5 \vee \neg x_6)[x_1, x_4, x_6] = \bot$.
- The subassignment $\{x_1, x_4\}$ can be repeated in other branches, the same clause is conflicting.
- Backjumping cannot avoid this problem.



Improvements to Backtracking Search: Backjumping is not optimal

Clause Learning (1/9)

Idea:

- A clause $(l_1 \vee l_2 \vee \ldots \vee l_k)$ is conflicting under the assignment $\{\neg l_1, \ldots \neg l_k\}$.
- QDPLL tries to satisfy clauses by unit literal detection in QBCP.
- Clauses in a PCNF guide QDPLL away from conflicts.
- Intuition(!): if a subassignment $A = \{l_1, l_2, \ldots, l_k\}$ is responsible for a conflict then add the clause $(\neg l_1 \vee \neg l_2 \vee \ldots \vee \neg l_k)$ to the PCNF.
- QDPLL tries to satisfy the added clause by assigning $\neg l_i$ for at least one l_i.
- QDPLL will not enumerate assignments A' such that $A \subseteq A'$.

Clause Learning (2/9)

Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6.(\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

- Every assignment A with $\{x_1, x_4\} \subseteq A$ is a conflict (under QBCP).
- Adding the clause $(\neg x_1 \vee \neg x_4)$ to ψ prevents QDPLL from repeating the subassignment $\{x_1, x_4\}$ in other branches.
- Assigning x_1 (x_4) triggers the assignment of $\neg x_4$ ($\neg x_1$) by unit literal detection in QBCP.

Clause Learning (3/9)

Properties:

- "clause learning" = adding clauses obtained from analyzing a conflict.
- "learned clause" = added clause.
- In general, adding arbitrary clauses to a PCNF ψ can make ψ unsatisfiable.
- Correctness of clause learning: $\psi \equiv \psi \wedge C$.

In Practice:

- Checking if $\psi \equiv \psi \wedge C$ is PSPACE-complete.
- How to efficiently find clauses C which can safely be added to ψ?

Resolution:

- Given the PCNF $\psi' = \psi \wedge C_1 \wedge C_2$, the resolution operation produces a new clause C_r (*resolvent*) from C_1 and C_2 such that $\psi' \equiv \psi' \wedge C_r$.
- By construction, a resolvent can safely be added to a PCNF.
- Idea: use resolution to produce learned clauses.

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- Combination of universal reduction and resolution for propositional logic.
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Definition (Q-Resolution)

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- Variable v is the *pivot* of the Q-resolution step.
- *Tentative Q-resolvent* of C_1 and C_2: $C_1 \otimes C_2 := (UR(C_1) \setminus \{v\}) \cup (UR(C_2) \setminus \{\neg v\})$.
- If $\{x, \neg x\} \subseteq C_1 \otimes C_2$ for some variable x, then no Q-resolvent exists.
- Otherwise, the non-tautological *Q-resolvent* is $C := UR(C_1 \otimes C_2)$.

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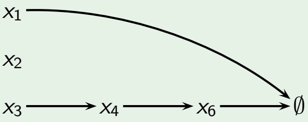
$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

- Conflict $A = \{x_1, x_2, x_3, x_4, x_6\}$, decisions x_1,x_2,x_3.
- Idea: consider antecedent clauses by unit literal detection and the conflicting clause for possible Q-resolutions, in reverse assignment order.
- Resolve $ante(\emptyset) = (\neg x_1 \vee y_5 \vee \neg x_6)$ and $ante(x_6) = (\neg x_4 \vee x_6)$, get tentative Q-resolvent $(\neg x_1 \vee \neg x_4 \vee y_5)$ and finally the Q-resolvent $(\neg x_1 \vee \neg x_4)$ by UR.
- Add $(\neg x_1 \vee \neg x_4)$ as a learned clause.



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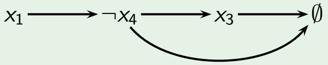
- Add $(\neg x_1 \vee \neg x_4)$ as a learned clause.
- Retract $\{x_2, x_3, x_4, x_6\}$, continue with $A = \{x_1\}$.
- By QBCP the learned clause $(\neg x_1 \vee \neg x_4)$ is unit and $A = \{x_1, \neg x_4\}$.
- Further, clause $(x_3 \vee x_4)$ is unit and $A = \{x_1, \neg x_4, x_3\}$.
- Conflict $A = \{x_1, \neg x_4, x_3\}$, clause $(\neg x_3 \vee x_4)[A] = \bot$ conflicting.
- Resolve $ante(\emptyset) = (\neg x_3 \vee x_4)$ and $ante(x_3) = (x_3 \vee x_4)$, get Q-resolvent (x_4).

After learning $(\neg x_1 \vee \neg x_4)$, continue with $A = \{x_1\}$:



Clause Learning (7/9)

Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

- Resolve $ante(\emptyset) = (\neg x_3 \vee x_4)$ and $ante(x_3) = (x_3 \vee x_4)$, get Q-resolvent (x_4).
- Add (x_4) as a learned clause.
- Retract $\{x_1, \neg x_4, x_3\}$, continue with $A = \{\}$.
- By QBCP, get $A = \{x_4, \neg x_1, x_6\}$ since the two learned clauses became unit.
- ...

After learning $(\neg x_1 \vee \neg x_4)$ and (x_4), continue with $A = \{\}$:

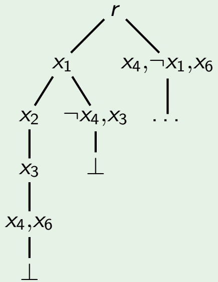
$$x_4 \longrightarrow \neg x_1 \longrightarrow x_6$$

Clause Learning (8/9)

Example (continued)

$\psi = \exists x_1, x_2, x_3, x_4 \forall y_5 \exists x_6 . (\neg x_3 \vee x_4) \wedge (x_3 \vee x_4) \wedge (\neg x_4 \vee x_6) \wedge (\neg x_1 \vee y_5 \vee \neg x_6) \wedge \phi.$

- Only three decisions in left branch: x_1, x_2, x_3.
- Other branches due to learned clauses which become unit after backtracking.
- Right branch: assignments $x_4, \neg x_1, x_6$ by unit literal detection due to learned clauses *without* decisions.
- Note: we never flipped decision variables explicitly.



Clause Learning (9/9)

Properties:

- Decisions are not explicitly flipped (unlike in backjumping).
- Our focus: learned clauses always become unit in QBCP after retracting assignments.
- Fundamentally different from backjumping and traditional backtracking.
- More powerful than backjumping: learned clauses prune search space.
- QDPLL learns the empty clause if and only if ψ is unsatisfiable.

Novel View on Search-Based Solving with Clause Learning:

- Assignment-driven engines searching for a Q-resolution proof (of unsatisfiable QBFs).
- Traditional backtracking view does not fit any more (also applies to SAT solvers).
- More appropriate name: *conflict-driven clause learning (CDCL)* for QBF (*QCDCL*).

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Combining QDPLL and Clause Learning

Modern Search-Based QBF Solving:

- Implementation: `analyze` must be adapted.
- Clause learning in `analyze`.
- Backtracking based on learned clause.
- No explicit flipping of decisions.
- Challenges: *efficient* implementation.

```
Result qdpll (PCNF f)
  Result r = UNDEF;
  Assignment a = {};
  while (true)
    /* Simplify. */
    (r,a) = qbcp (f,a);
    if (r == UNDET)
      /* Decision making. */
      a = assign_dec_var (f,a);
    else
      /* Backtracking. */
      /* r == UNSAT or r == SAT */
      btlevel = analyze (r,a);
      if (btlevel == INVALID)
        return r;
      else
        a = backtrack (btlevel);
```

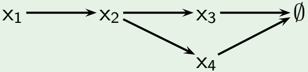
- *In reverse assignment order*, resolve on existential variables which were assigned as unit literals, using clauses (i.e. antecedents) which became unit during QBCP.
- Tautological resolvents by universal literals might occur but must be avoided: deviate from strict reverse assignment order [GNT06].
- Worst case exponential number of intermediate resolvents [VG12].

Example

$$\exists x_1, x_3, x_4 \forall y_5 \exists x_2$$
$$(\neg x_1 \vee x_2) \wedge$$
$$(x_3 \vee y_5 \vee \neg x_2) \wedge$$
$$(x_4 \vee \neg y_5 \vee \neg x_2) \wedge$$
$$(\neg x_3 \vee \neg x_4)$$

Assignment $A = \{x_1, x_2, x_3, x_4\}$
Assignment order: x_1, x_2, x_3, x_4
Can we resolve in reverse assignment order?

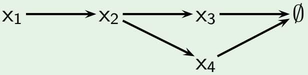
Clause $(\neg x_3 \vee \neg x_4)$ conflicting:



Example

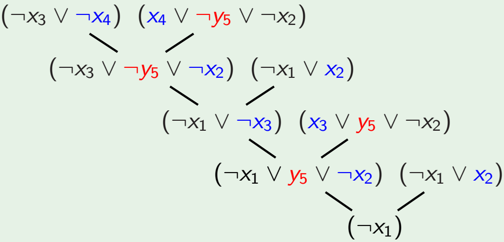
$$\exists x_1, x_3, x_4 \forall y_5 \exists x_2$$
$$(\neg x_1 \lor x_2) \land$$
$$(x_3 \lor y_5 \lor \neg x_2) \land$$
$$(x_4 \lor \neg y_5 \lor \neg x_2) \land$$
$$(\neg x_3 \lor \neg x_4)$$

Clause $(\neg x_3 \lor \neg x_4)$ conflicting:



Assignment $A = \{x_1, x_2, x_3, x_4\}$
Assignment order: x_1, x_2, x_3, x_4
Resolve on: x_4, x_2 (!), x_3, x_2

Derivation of learned clause $(\neg x_1)$:



- Linear-time procedure: resolve *in* assignment order [LEVG13].

Towards a Proof System for PCNFs

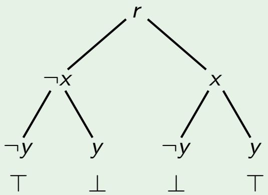
Learning from Solutions (1/4)

Cube Learning:

- Given a solution A, i.e. $\psi[A] = \top$.
- Solution A is a branch in the assignment tree with a $\top$-leaf.
- All clauses in ψ are satisfied under A.
- Idea: record A as a conjunction of literals: *learned cube*.
- Dual to clauses, learned cubes become unit in QBCP after backtracking and prevent the solver from enumerating the same subassignment.

Example

- Satisfiable PCNF $\psi = \forall x \exists y.(x \lor \neg y) \land (\neg x \lor y)$.



Definition (model generation rule [GNT06])

Given a PCNF $\psi := \hat{Q}.\phi$ and a solution A, i.e. $\psi[A] = \top$. An *initial cube* $C = (\bigwedge_{l_i \in A} l_i)$ is a conjunction over the literals of a solution A.

Example

$\psi := \exists x_1 \forall y_8 \exists x_5, x_2, x_6, x_4. (y_8 \vee \neg x_5) \wedge (x_2 \vee \neg x_6) \wedge (\neg x_1 \vee x_4) \wedge (\neg y_8 \vee \neg x_4) \wedge (x_1 \vee x_6) \wedge (x_4 \vee x_5)$.

Solution $A_1 := \{x_6, x_2, \neg y_8, \neg x_5, x_4\}$, initial cube $C_1 := (x_6 \wedge x_2 \wedge \neg y_8 \wedge \neg x_5 \wedge x_4)$.

Solution $A_2 := \{y_8, \neg x_4, \neg x_1, x_5, x_6, x_2\}$, initial cube $C_2 := (y_8 \wedge \neg x_4 \wedge \neg x_1 \wedge x_5 \wedge x_6 \wedge x_2)$.

Learning from Solutions (3/4)

Definition

Given a cube C, *existential reduction (ER)* on C produces the cube

$$ER(C) := C \setminus \{l \in C \mid q(l) = \exists \text{ and } \forall l' \in C \text{ with } q(l') = \forall : var(l') < var(l)\},$$

where $<$ is the linear variable ordering given by the quantifier prefix.

- ER is dual to universal reduction, deletes "trailing" existential literals from cubes.
- ER shortens cubes.

Example (continued)

$\psi := \exists x_1 \forall y_8 \exists x_5, x_2, x_6, x_4. (y_8 \vee \neg x_5) \wedge (x_2 \vee \neg x_6) \wedge (\neg x_1 \vee x_4) \wedge (\neg y_8 \vee \neg x_4) \wedge (x_1 \vee x_6) \wedge (x_4 \vee x_5)$.

Initial cube $C_1 := (x_6 \wedge x_2 \wedge \neg y_8 \wedge \neg x_5 \wedge x_4)$.

$C_3 := ER(C_1) = (\neg y_8)$

Initial cube $C_2 := (y_8 \wedge \neg x_4 \wedge \neg x_1 \wedge x_5 \wedge x_6 \wedge x_2)$.

$C_4 := ER(C_2) = (y_8 \wedge \neg x_1)$

Definition (cube resolution [GNT06, ZM02])

Given two non-contradictory cubes C_1 and C_2, *cube resolution* is defined analogously to Q-resolution for clauses, except:

- existential reduction.
- universal variables as pivots.

The cube resolvent of C_1 and C_2 (if it exists) is denoted by $C := C_1 \otimes C_2$.

Example (continued)

$\psi := \exists x_1 \forall y_8 \exists x_5, x_2, x_6, x_4. (y_8 \vee \neg x_5) \wedge (x_2 \vee \neg x_6) \wedge (\neg x_1 \vee x_4) \wedge (\neg y_8 \vee \neg x_4) \wedge (x_1 \vee x_6) \wedge (x_4 \vee x_5)$.

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$C_4 := ER(C_2) = (y_8 \wedge \neg x_1)$

$C_5 := C_3 \otimes C_4 = (\neg x_1)$, $ER(C_5) = \emptyset$.

QCDCL as a Proof System

Cube Learning:

- Model generation, existential reduction, cube resolution.
- The empty cube is derived if and only if the PCNF satisfiable.
- Dual to clause learning: driven by assignment generation, implication graphs.

Clause Learning:

- Universal reduction, Q-resolution.
- The empty clause is derived if and only if the PCNF unsatisfiable.

Definition

- PCNF $\psi := \hat{Q}.\,\phi$ with quantifier prefix $\hat{Q}$ and CNF ϕ.
- *Augmented CNF* of ψ: $\psi' := \hat{Q}.\,(\phi \wedge \theta \vee \gamma)$.
- Original clauses ϕ.
- Learned clauses θ, filled during clause learning.
- Learned cubes γ, filled during cube learning.
- Properties: $\hat{Q}.\,\phi \equiv \hat{Q}.\,(\phi \wedge \theta)$ and $\hat{Q}.\,\phi \equiv \hat{Q}.\,(\phi \vee \gamma)$.

Final View: QCDCL $\neq$ QDPLL + Learning

- QCDCL generates proofs.
- Proof generation is driven by assignments.
- QCDCL does not flip decision variables explicitly.
- Backtracking is driven by learned clauses and cubes.
- Problem: CNF is bad for cube learning.

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Optimizations

Inspired by efficient solvers for propositional logic (SAT).

Restarts:

- Periodically retract all assignments and start over with $A = \{\}$.
- Makes the solver incomplete unless restart period grows sufficiently large.
- Idea: getting out of "bad" regions in the search space.

Assignment Caching:

- Store assigned values other than decisions in a per-variable cache.
- If variable x is selected to make a decision, then assign cached value of x.
- Idea: re-use previous assignments in similar parts of the formula.

Deletion of Learned Clauses and Cubes:

- Formula grows steadily by addition of learned clauses and cubes.
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- Tautological resolvents C, i.e. where $\{v, \neg v\} \subseteq C$, are disallowed.
- In general, tautological resolvents might produce unsound results.

Long-Distance (LD) Resolution:

- Allow to produce *certain* tautological resolvents: soundness.
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- Implemented in yQuaffle, DepQBF for clause learning.

QU-Resolution:

- Allow to resolve over universally quantified variables.
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Future Work:

- How to integrate QU-resolution systematically into QCDCL?

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Challenge: Variable Dependencies (1/2)

Order of Assignments:

- Given the PCNF $Q_1 B_1 Q_2 B_2 \dots Q_m B_m.\phi$, QDPLL in general must only assign variables as decisions starting from B_1 to ensure soundness.

Example

The PCNF $\psi = \forall x \exists y.(x \vee \neg y) \wedge (\neg x \vee y)$ is satisfiable.
The PCNF $\psi = \exists y \forall x.(x \vee \neg y) \wedge (\neg x \vee y)$ is unsatisfiable.

- This linear ordering limits the freedom to select decision variables.

Example

$\psi = Q_1 B_1 \dots Q_n B_n.\phi \wedge \phi'(B_n)$.

- ϕ: hard formula.
- $\phi'(B_n)$: easy formula over variables in B_n.
- QDPLL tends to assign variables in B_n late (except in QBCP).
- QDPLL attempts to solve hard ϕ first.

Dependency Analysis:

- Are there variables which can be moved to the left end in the quantifier prefix without changing the satisfiability of ψ?
- Related work: quantifier shifting / miniscoping in theorem proving.
- PSPACE-complete problem.

Dependency Schemes:

- Binary relation $D \subseteq V \times V$ over variables in a PCNF.
- If $(x, y) \in D$ then must assign x before y to ensure soundness.
- If $(x, y) \notin D$ then can assign x before y or vice versa.
- D computed by a syntactic analysis of the PCNF.
- Tradeoff: efficiency of computation and precision.
- Dependency Schemes in DepQBF: efficient integration as compact graphs.

Challenge: Variable Dependencies (2/2)

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Recent Trends

Preprocessing:

- Impressive reduction in formula size and solving time.
- Can be harmful for solvers relying on formula structure.
- Applications: preprocessing can destroy the original encoding (certificates).

CNF-based Solving and Structure Reconstruction:

- Dedicated non-PCNF solvers operating on e.g. circuit structure.
- Recent focus on CNF data structures due to efficiency.
- Structure reconstruction to extract circuit information from a CNF.
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Summary

Search-Based QBF Solving (QDPLL):

- Originates from backtracking search (1960s), like SAT solvers.
- Modern implementations: fundamentally different.

QCDCL: Assignment Generation + Clause/Cube Learning:

- More powerful than backtracking/backjumping.
- No explicit flipping of decision variables.
- Generation of resolution proofs guided by assignments.
- State-of-the-art approach, crucial implementation details.
- Future work: safely relax the quantifier ordering.
- Future work: integrate preprocessing, certificate generation.

`http://lonsing.github.io/depqbf/`

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