# SAT-BASED BOUNDED MODEL CHECKING

## Formal Models SS19

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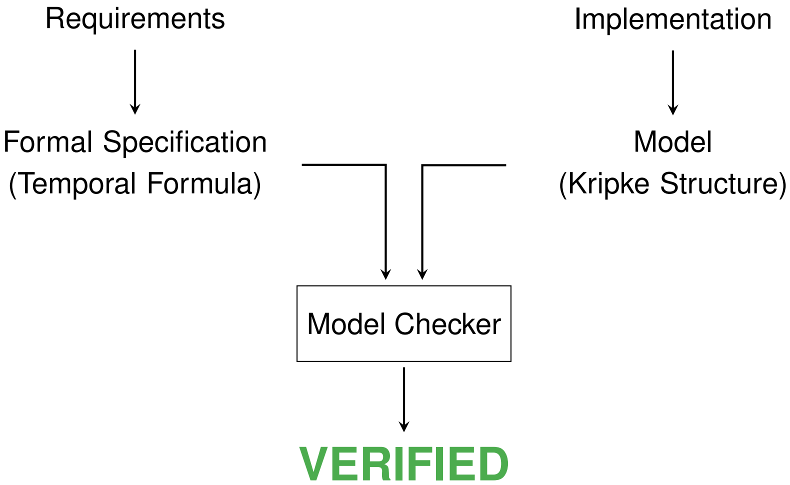
# Model Checking



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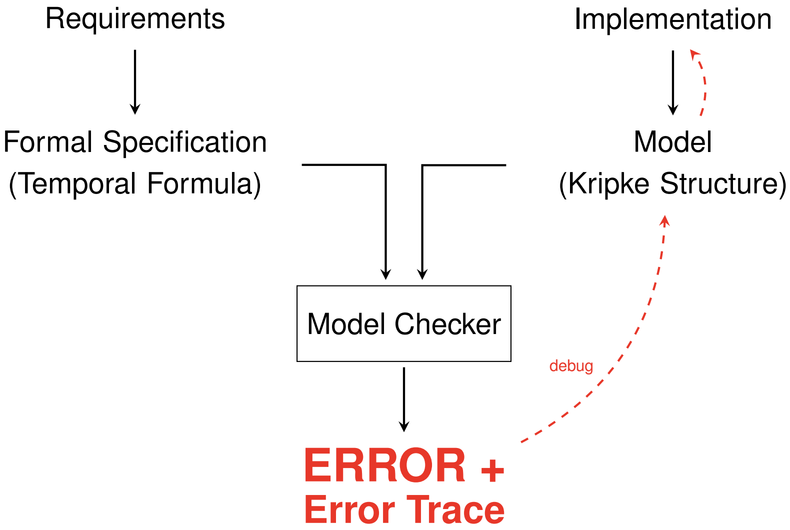


# Types of Model Checking

**General question**: Given a system $K$ and a property $p$, does $p$ hold for $K$ (i.e., for all initial states of $K$) ?

- Explicit state model checking
  - enumeration of the state space
  - state explosion problem
- Symbolic model checking
  - representation of model checking problem as logical formula (e.g., in propositional logic (SAT) or QBF)

# Some Properties

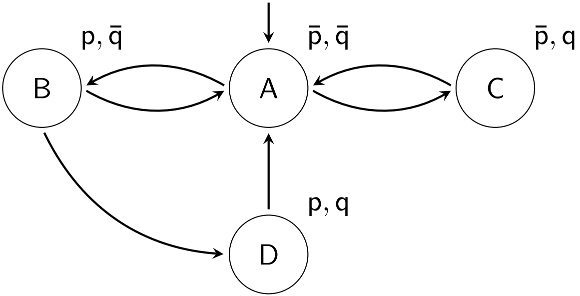
- **Reachability**: property $p$ holds in one reachable state
- **Invariant**: property $p$ holds in all reachable states
- **Safety**: some bad property $p$ never holds
  "something bad will never happen"
- **Liveness**: something good will eventually happen
- **Fairness**: under certain conditions, some property holds repeatedly

# Example: Mutual Exclusion

Given two processes P and Q which share a resource R.

- If R is accessed by P, then property p is true.
- If R is accessed by Q, then property q is true.

The behavior of P and Q is modeled by this Kripke structure:




**Question**: Does $F(p \wedge q)$ hold?

# Limboole

- SAT-solver for formulas in non-CNF
- available at `http://fmv.jku.at/limboole/`
- input format in BNF:

$$
\begin{aligned}
\langle expr \rangle &::= \langle iff \rangle \\
\langle iff \rangle &::= \langle implies \rangle \mid \langle implies \rangle \text{ “}\texttt{<->}\text{” } \langle implies \rangle \\
\langle implies \rangle &::= \langle or \rangle \mid \langle or \rangle \text{ “}\texttt{->}\text{” } \langle or \rangle \mid \langle or \rangle \text{ “}\texttt{<-}\text{” } \langle or \rangle \\
\langle or \rangle &::= \langle and \rangle \mid \langle and \rangle \text{ “}\texttt{|}\text{” } \langle and \rangle \\
\langle and \rangle &::= \langle not \rangle \mid \langle not \rangle \text{ “}\texttt{\&}\text{” } \langle not \rangle \\
\langle not \rangle &::= \langle basic \rangle \mid \text{“}\texttt{!}\text{” } \langle not \rangle \\
\langle basic \rangle &::= \langle var \rangle \mid \text{“}\texttt{(}\text{” } \langle expr \rangle \text{ “}\texttt{)}\text{”}
\end{aligned}
$$

where ’var’ is a string over letters, digits, and

`- _ . [ ] $ @`

# Symbolic Encoding of Kripke Structures

Given Kripke structure $K = (S, I, T, L)$ over $\mathcal{A} = \{a_1, \ldots, a_n\}$.

1. Introduce sets $\mathcal{A}' = \{a'_1, \ldots, a'_n\}$ and $\mathcal{A}'' = \{a''_1, \ldots, a''_n\}$ for the **definition of one transition step $\mathcal{T}$ over $\mathcal{A}'$ and $\mathcal{A}''$**.
2. Associate each state $s \in S$ with two conjunctions of literals $current(s)$ and $next(s)$:[1]
   - $current(s) := (l_1 \wedge \ldots \wedge l_n)$ such that $l_i = a'_i$ if $a_i \in L(s)$ else $l_i = \bar{a}'_i$;
   - $next(s) := (k_1 \wedge \ldots \wedge k_n)$ such that $k_i = a''_i$ if $a_i \in L(s)$ else $k_i = \bar{a}''_i$.
3. Define prop. formula $\mathcal{T}$ over $\mathcal{A}', \mathcal{A}''$ such that $\forall s_i, s_j \in S$ $(\mathcal{T} \wedge current(s_i) \wedge next(s_j))$ **is satisfiable iff** $(s_i, s_j) \in T$.

---

[1] note: the mapping "state to conjunction" has to be bijective

# Naive Encoding of Kripke Structures in SAT

Let $K = (S, I, T, L)$ be a Kripke structure over $\mathcal{A}$.

$\mathcal{T} := \top$

**while** $S \neq \emptyset$ **do**

  select $s \in S$

  $S := S \backslash \{s\}$

  $N := \bot$

  **for all** $(s, t) \in T$ **do**

    $N := N \vee next(t)$

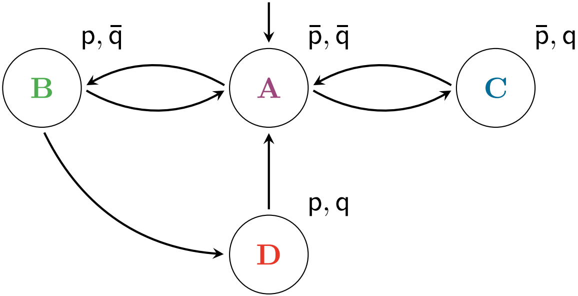
  **end for**

  $\mathcal{T} := \mathcal{T} \wedge (current(s) \rightarrow N)$

**end while**

**return** $\mathcal{T}$

# Naive Encoding of Kripke Structures in SAT




$$
\begin{array}{rcll}
\mathcal{T} & := & \top & \wedge \\
& & (\bar{p} \wedge \bar{q}) \rightarrow (\bot \vee (\bar{p}' \wedge q') \vee (p' \wedge \bar{q}')) & \wedge \\
& & (p \wedge \bar{q}) \rightarrow (\bot \vee (\bar{p}' \wedge \bar{q}') \vee (p' \wedge q')) & \wedge \\
& & (\bar{p} \wedge q) \rightarrow (\bot \vee (\bar{p}' \wedge \bar{q}')) & \wedge \\
& & (p \wedge q) \rightarrow (\bot \vee (\bar{p}' \wedge \bar{q}')) &
\end{array}
$$

# Naive Encoding of Kripke Structures in SAT

Encoding in Limboole syntax:

```
((!p & !q)  -> (!p-next & q-next) | (p-next & !q-next)) &
((p & !q)   -> (!p-next & !q-next) | (p-next & q-next)) &
((!p & q)   -> (!p-next & !q-next)) &
((p & q)   -> (!p-next & !q-next))
```

```
> limboole limboole/mutual.boole -s
% SATISFIABLE formula (satisfying assignment follows)

p = 0
q = 0
p-next = 0
q-next = 1
```

# Symbolic Encoding of Kripke Structures

Alternative encoding of transition function:

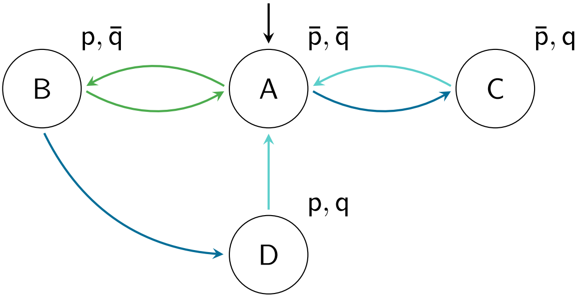
**Successor states** $p', q'$:

$(p' \leftrightarrow (\bar{p} \wedge \bar{q}) \wedge q' \leftrightarrow 0)$

$\vee$

$(p' \leftrightarrow (p \wedge \bar{q}) \wedge q' \leftrightarrow \bar{q})$

| p | q | p′ | q′ | or | p′ | q′ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | | 0 | 1 |
| 0 | 1 | 0 | 0 | | 0 | 0 |
| 1 | 0 | 0 | 0 | | 1 | 1 |
| 1 | 1 | 0 | 0 | | 0 | 0 |



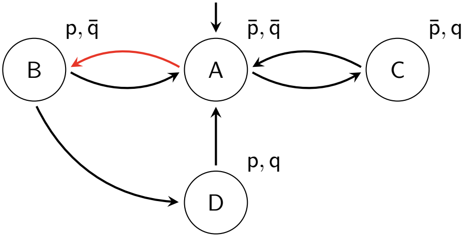
# Example: One Step

Encoding in Limboole syntax:

```
(((p-next <-> (!p & !q)) & (!q-next)) |
((p-next <-> (p & !q)) & (q-next <-> !q)))

> limboole -s mutual2.boole
% SATISFIABLE formula (satisfying assignment follows)

p = 0
q = 0
p-next = 1
q-next = 0
```



# Multiple Transition Steps

- $\mathcal{T}$ over $\mathcal{A}'$ and $\mathcal{A}''$ defines one transition step
  - we also write $\mathcal{T}(s_0, s_1)$ indicating that we can go from state $s_0$ to a state $s_1$
- $\mathcal{T}$ over $\mathcal{A}''$ and $\mathcal{A}'''$ defines one transition step
  - we also write $\mathcal{T}(s_1, s_2)$ indicating that we can go from state a $s_1$ to a state $s_2$
- $\mathcal{T}(s_0, s_1) \wedge \mathcal{T}(s_1, s_2)$ defines two transition steps from a state $s_0$ to a state $s_1$
- Example (previous slides):

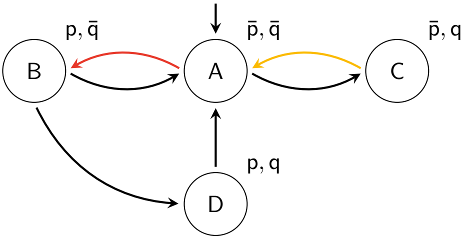
$$(((\mathsf{p}' \leftrightarrow (\bar{\mathsf{p}} \wedge \bar{\mathsf{q}})) \wedge (\mathsf{q}' \leftrightarrow 0)) \vee ((\mathsf{p}' \leftrightarrow (\mathsf{p} \wedge \bar{\mathsf{q}})) \wedge (\mathsf{q}' \leftrightarrow \bar{\mathsf{q}}))) \quad \wedge$$
$$(((\mathsf{p}'' \leftrightarrow (\bar{\mathsf{p}}' \wedge \bar{\mathsf{q}}')) \wedge (\mathsf{q}'' \leftrightarrow 0)) \vee ((\mathsf{p}'' \leftrightarrow (\mathsf{p}' \wedge \bar{\mathsf{q}}')) \wedge (\mathsf{q}'' \leftrightarrow \bar{\mathsf{q}}')))$$

# Example: Two Steps

Encoding in Limboole syntax:

```
((((p-next <-> (!p & !q)) & (!q-next))  |
((p-next <-> (p & !q)) & (q-next <-> !q)))) &
((((p-next2 <-> (!p-next & !q-next)) & (!q-next2)) |
((p-next2 <-> (p-next & !q-next)) & (q-next2 <-> !q-next))))
```

```
> limboole -s mutual2-twoSteps.boole
% SATISFIABLE formula (satisfying assignment follows)
p = 0
q = 1
p-next = 0
q-next = 0
p-next2 = 1
q-next2 = 0
```

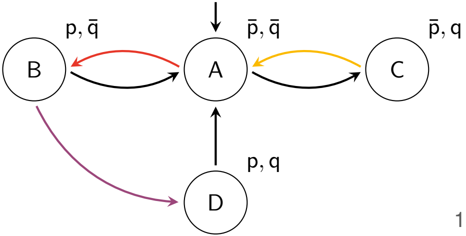


# Example: Three Steps

Encoding in Limboole syntax:

```
((((p-next <-> (!p & !q)) & (!q-next))  |
((p-next <-> (p & !q)) & (q-next <-> !q)))) &
((((p-next2 <-> (!p-next & !q-next)) & (!q-next2)) |
((p-next2 <-> (p-next & !q-next)) & (q-next2 <-> !q-next)))) &
((((p-next3 <-> (!p-next2 & !q-next2)) & (!q-next3)) |
((p-next3 <-> (p-next2 & !q-next2)) & (q-next3 <-> !q-next2))))
```

```
limboole -s mutual2-threeSteps.boole
% SATISFIABLE formula (satisfying assignment follows)
p = 0
q = 1
p-next = 0
q-next = 0
p-next2 = 1
q-next2 = 0
p-next3 = 1
q-next3 = 1
```



# Bounded Model Checking (Safety)

- Given a Kripke structure $K$. Is there a path of length $k$ to a **bad state** $s$, i.e., a certain property $p$ is violated in $s$?
- In other words: there is a path where $\mathsf{G}p$ does not hold in $K$
- Observation: if $\mathsf{G}p$ does not hold in $K$, there is a **finite counter-example.**
- Idea: consider paths of fixed length $k$
  - encode problem to propositional formula $\phi$
  - pass problem to SAT solver
  - $\phi$ is true $\Leftrightarrow$ model of $\phi$ is counter-example
  - if $\phi$ is false, then increase $k$

# Bounded Model Checking (Safety)

A bounded model checking (BMC) problem for Kripke structure $K$ and safety property $\mathsf{G}p$ is encoded by

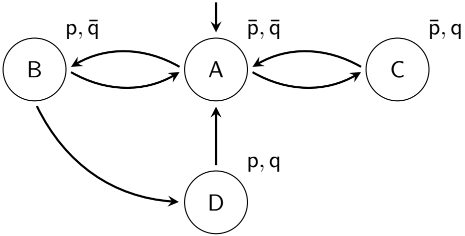
$$I(s_0) \wedge \mathcal{T}(s_0, s_1) \wedge \mathcal{T}(s_1, s_2) \wedge \ldots \wedge \mathcal{T}(s_{k-1}, s_k) \wedge B(s_k)$$

where

- $I(s_0)$ is true $\Leftrightarrow$ $s_0$ is an initial state
- $\mathcal{T}$ is the transition function of $K$
- $B(s_k)$ is true $\Leftrightarrow$ $s_k$ is a bad state, i.e., $\neg p$ holds in $s_k$

# BMC Example

We want to know if $\mathsf{G}(\bar{\mathsf{p}} \vee \bar{\mathsf{q}})$ holds for Kripke structure $K$:



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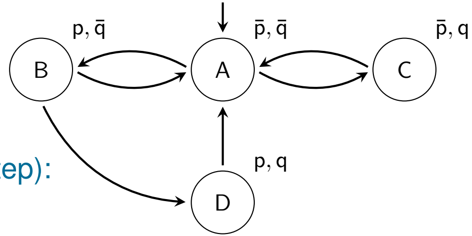


$k = 1$ (one step):

$(\bar{\mathsf{p}} \wedge \bar{\mathsf{q}}) \quad \wedge$

$((\mathsf{p}' \leftrightarrow (\bar{\mathsf{p}} \wedge \bar{\mathsf{q}})) \wedge (\mathsf{q}' \leftrightarrow 0)) \vee ((\mathsf{p}' \leftrightarrow (\mathsf{p} \wedge \bar{\mathsf{q}})) \wedge (\mathsf{q}' \leftrightarrow \bar{\mathsf{q}})) \quad \wedge$

$(\mathsf{p}' \wedge \mathsf{q}')$

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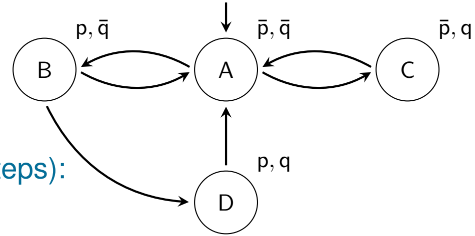


$k = 2$ (two steps):

$(\bar{p} \wedge \bar{q}) \quad \wedge$

$(((p' \leftrightarrow (\bar{p} \wedge \bar{q})) \wedge (q' \leftrightarrow 0)) \vee ((p' \leftrightarrow (p \wedge \bar{q})) \wedge (q' \leftrightarrow \bar{q})) \quad \wedge$

$(((p'' \leftrightarrow (\bar{p}' \wedge \bar{q}')) \wedge (q'' \leftrightarrow 0)) \vee$

$((p'' \leftrightarrow (p' \wedge \bar{q}')) \wedge (q'' \leftrightarrow \bar{q}'))) \quad \wedge$

$(p'' \wedge q'')$

# Compact BMC Encoding with QBF

A bounded model checking (BMC) problem for Kripke structure $K$ and property $\mathsf{G}p$ is encoded by

$$\exists s_0, s_1, \ldots, s_k \forall x, x'. \quad (I(s_0) \quad \wedge \quad B(s_k) \quad \wedge \\ (\bigvee_{i=0}^{k-1} (x \leftrightarrow s_i \wedge x' \leftrightarrow s_{i+1}) \rightarrow \mathcal{T}(x, x')))$$

where

- $I(s_0)$ is true $\Leftrightarrow$ $s_0$ is an initial state
- $\mathcal{T}$ is the transition relation of $K$
- $B(s_k)$ is true $\Leftrightarrow$ $s_k$ is a bad state, i.e., $p$ holds in $s_k$

**Advantage: only one copy of transition relation!**

# Quantified Boolean Formulas (QBF)

- Extension of propositional logic
  - explicit quantifiers ($\forall$, $\exists$) over the Boolean variables
- Canonical PSPACE-complete problem
  - more succinct encoding than SAT (NP-complete)
- Many application domains: synthesis, AI, verification, ...

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$$\overbrace{\exists x \exists y \forall u \exists z . (u \rightarrow z) \wedge (y \vee u \vee \neg z) \wedge (x \vee \neg u \vee \neg z) \wedge (x \leftrightarrow \neg y)}^{\text{closed QBF in prenex form}}$$

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$$\overbrace{\underbrace{\exists x \exists y \forall u \exists z.}_{\text{prefix}}\underbrace{(u \rightarrow z) \wedge (y \vee u \vee \neg z) \wedge (x \vee \neg u \vee \neg z) \wedge (x \leftrightarrow \neg y)}_{\text{matrix}}}^{\text{closed QBF in prenex form}}$$

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# QBF Syntax

- **QBFs in Prenex CNF (PCNF):**

$$\exists x \exists y \forall u \exists z . \underbrace{(\overset{\text{literals}}{\neg u \vee z}) \wedge \overbrace{(y \vee u \vee \neg z)}^{\text{clause}} \wedge (x \vee \neg u \vee \neg z)}_{\text{CNF}}$$

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- **QBFs in Prenex DNF (PDNF):**

$$\forall x \forall y \exists u \forall z.\underbrace{\overbrace{(u \wedge \neg z)}^{\text{cube}} \vee (\neg y \wedge \neg u \wedge z) \vee (\neg x \wedge u \wedge z)}_{\text{DNF}}$$

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**Note:** $x, y < u < z$

# QBF Semantics

- $\forall x \mathcal{Q}.\varphi$ true $\Leftrightarrow$ $\mathcal{Q}.\varphi[x]$ **and** $\mathcal{Q}.\varphi[\neg x]$ true

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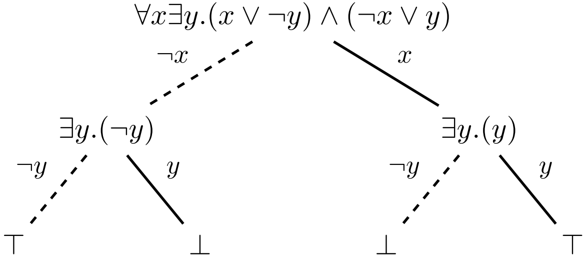
- Example:

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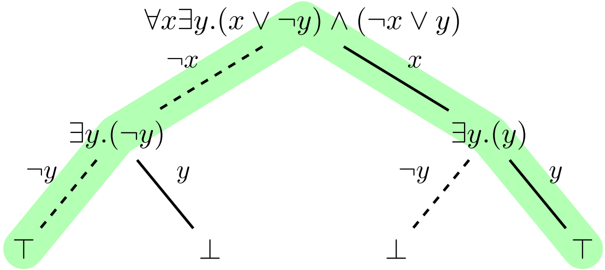
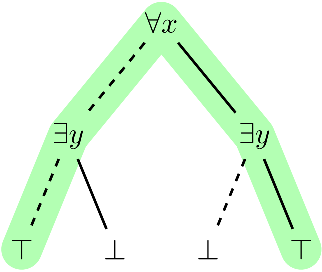


# QBF Models

Tree model of **true** formula:

$$\forall x \exists y.(x \vee \bar{y}) \wedge (\bar{x} \vee y)$$



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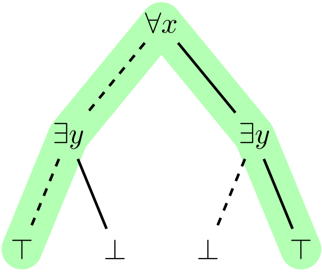
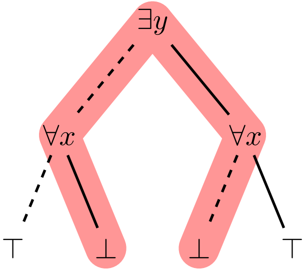



Tree refutation of **false** formula:

$$\exists y \forall x . (x \vee \bar{y}) \wedge (\bar{x} \vee y)$$



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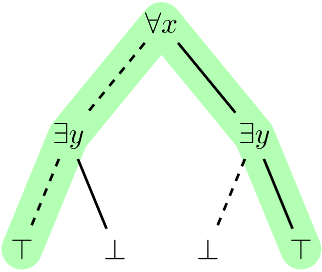


Skolem-functions of

$\exists$-variables:

$$f_y(x) = x$$

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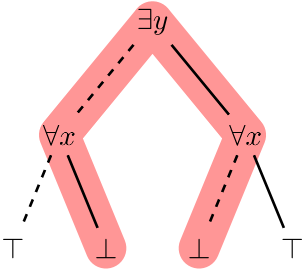

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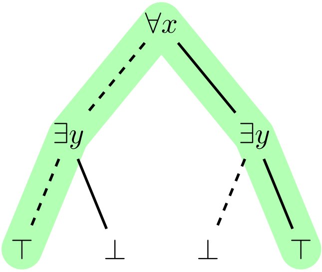


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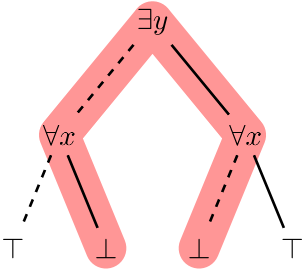


Herbrand-functions of

$\forall$-variables:

$$f_x(y) = \bar{y}$$

```
Boolean splitCNF (Prefix $P$, matrix $\psi$)

if ($\psi == \emptyset$): return true;
if ($\emptyset \in \psi$): return false;

$P = QXP', x \in X, \;\; X' = X \backslash \{x\}$;

if ($Q == \forall$)
     return (splitCNF($QX'P', \psi'$) &&
             splitCNF($QX'P', \psi''$));
else
     return (splitCNF($QX'P', \psi'$) ||
             splitCNF($QX'P', \psi''$));
where
$\psi' : take\ clauses\ of\ \psi,\ delete\ clauses\ with\ x,\ delete\ \neg x$
$\psi'' : take\ clauses\ of\ \psi,\ delete\ clauses\ with\ \neg x,\ delete\ x$
```

# BMC Summary

- BMC is incomplete ...
  - if all checked formulas are unsat, no insight
  - how to choose $k$? when to stop increasing $k$?
- ... very efficient (e.g., debugging)
- many tuning techniques
  - exploit similarities between two transition steps (structure sharing)
  - simplification of formula by rewritings )