Open Problems for Quantified Boolean Formulas

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Introduction

- Fixed Deficiency for QCNF
- QHorn and Satisfiabilty
- Related Horn Problems
    1. Equivalence Problem
    2. Literal Problem
- Expressive Power
    1. Equivalence Models and Propositional Formulas
    2. QHorn and Q2-CNF
    3. Model size and Deficiency

Formula $\alpha = \alpha_1 \wedge \ldots \alpha_m \in$CNF over the variables $x_1, \ldots, x_n$.
α is **minimal unsatisfiable (MU)** iff $\alpha \in \overline{\text{SAT}}$ and $\alpha \setminus \alpha_i \in \text{SAT}$ for every i.
The **deficiency** is defined as $d(\alpha) = m - n$.
MU(k) is the set of MU-formulas with deficiency k

Theorem

1. *MU is D^P-complete.* *(SAT, $\overline{\text{SAT}}$, Papadimitriou, Wolf)*
2. *Every formula in MU has deficiency greater than 0 (Lional at all).*
3. *Every minimal unsatisfiable Horn formula has deficiency 1.*
4. *MU(k) is solvable in polynomial time. (Kullmann, Szeider)*

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QCNF: Quantified Boolean formulas with kernel in CNF

Example: $\Phi = \forall x \exists y : (x \vee y) \wedge (x \vee \neg y)$

Φ is false, but
$\Phi \setminus (x \vee y) = \forall x \exists y : (x \vee \neg y)$ and
$\Phi \setminus (x \vee \neg y) = \forall x \exists y : (x \vee \neg y)$ are true.
Φ is **minimal false**.

Deficiency: number of clauses - number of existential variables

$d(\Phi) = 2 - 1 = 1$

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Extension to closed QCNF (minimal falsity and deficiency)

Let $\Phi = Q \bigwedge_{1 \leq i \leq n} \varphi_i \in$ QCNF with universal variables $x_1, \ldots, x_t$ and existential variables $y_1, \ldots, y_r$.

Definition

1. The formula Φ is **minimal false (MF)** iff Φ is false and for every j the formula $Q \bigwedge_{1 \leq i \neq j \leq n} \varphi_i$ is true.
2. The **deficieny** is defined as $d(\Phi) = n - |var(\varphi_{|\exists})|$.
(number of clauses minus the number the existential variables)
3. For fixed k we define $\mathbf{MF(k)} = \{\Phi : \Phi \in MF \text{ and } d(\Phi) = k\}$.

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Theorem

(KB, Zhao)

1. *The minimal falsity problem MF is PSPACE-complete.*
2. *If* $\Phi \in MF$, *then* $d(\Phi) \geq 1$.
3. $MF(1)$ *is solvable in polynomial time.*
4. *For fixed* $k \geq 1$: $MF(k)$ *is in* D^P.

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4. *For fixed* $k \geq 1 : MF(k)$ *is in* D^P.

Open Problem:The computational complexity of $MF(k)$ for fixed $k \geq 2$.

DHorn: conjunction of implications $(a_1, \ldots, a_n \rightarrow b)$ and facts (a)
Horn: DHorn $\cup$ negative clauses $(\neg a_1 \vee \ldots \vee \neg a_n)$

QHorn: set of quantified Boolean formulas in prenex normal form with matrix in Horn.
Let $\Phi = Q\phi \in$ QHorn without free variables and prefix Q.
Let k be the number of universal quantifiers.

Theorem

(KB at all)
The satisfiability problem is solvable in time $O(k \cdot |\Phi|)$.

Open Problem: Can we solve the satisfiability problem for QHorn in linear time?

QHorn and Multi-Horn-SAT

Multi-Horn-SAT:

Instance: $\alpha \in$ DHorn, $r \geq 1$, $Y = \{y_1, \ldots, y_m\}, S_1, \ldots, S_r \subseteq Y$, $N_1, \ldots, N_r$ negative clauses

Query: $\exists j : S_j \wedge \alpha \wedge N_j \in \overline{\text{SAT}}$?

y_1			y_1
		y_2	y_2
	y_3	y_3	
y_4	y_4		
α	α	α	α
N_1	N_2	N_3	N_4

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QHorn

Instance: $\alpha \in$ DHorn, $r \geq 1$, $Y = \{y_1, \ldots, y_m\}$, $S_1, \ldots, S_r \subseteq Y$, $N_1, \ldots, N_r$ negative clauses
Query: $\exists j : S_j \wedge \alpha \wedge N_j \in \overline{\text{SAT}}$?

y_1			y_1		$\neg x_2 y_1^1$	$\neg x_3 \neg y_1^1 y_1$	
		y_2	y_2	$\neg x_1 y_2^1$	$\neg x_2 \neg y_2^1 y_2$		
	y_3	y_3		$\neg x_1 y_3^1$			$\neg x_4 \neg y_3^1 y_3$
y_4	y_4					$\neg x_3 y_4^1$	$\neg x_4 \neg y_4^1 y_4$
α	α	α	α	α	α	α	α
N_1	N_2	N_3	N_4	$x_1 N_1$	$x_2 N_2$	$x_3 N_3$	$x_4 N_4$

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y_1			y_1		$\neg x_2 y_1^1$	$\neg x_3 \neg y_1^1 y_1$	
		y_2	y_2	$\neg x_1 y_2^1$	$\neg x_2 \neg y_2^1 y_2$		
	y_3	y_3		$\neg x_1 y_3^1$			$\neg x_4 \neg y_3^1 y_3$
y_4	y_4					$\neg x_3 y_4^1$	$\neg x_4 \neg y_4^1 y_4$
α	α	α	α	α	α	α	α
N_1	N_2	N_3	N_4	$x_1 N_1$	$x_2 N_2$	$x_3 N_3$	$x_4 N_4$

$$\forall x_1 \forall x_2 \forall x_3 \forall x_4 \exists Y : \alpha,\ \bigwedge_{1 \leq i \leq 4} (x_i \vee N_i),$$
$$(\neg x_2 \vee y_1^1), (\neg x_3 \vee \neg y_1^1 \vee y_1), (\neg x_1 \vee y_2^1), (\neg x_2 \vee \neg y_2^1 \vee y_2),$$
$$(\neg x_1 \vee y_3^1), (\neg x_4 \vee \neg y_3^1 \vee y_3), (\neg x_3 \vee y_4^1), (\neg x_4 \vee \neg y_4^1 \vee y_4)$$

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Related Horn Problems

1. Instance: $\alpha, \beta \in$ Horn
   Query: (Equivalence) $\alpha \approx \beta$?
   Solvable in quadratic time
   Open problem: Solvable in linear or $O(n\ log(n))$ time?

Related Horn Problems

1. Instance: $\alpha, \beta \in$ Horn
   Query: (Equivalence) $\alpha \approx \beta$?
   Solvable in quadratic time
   Open problem: Solvable in linear or $O(n\ log(n))$ time?

2. Instance: $\alpha \in$ Horn over the variables $X = \{x_1, \ldots, x_m\}$
   Query: Compute $\mathrm{NL}(\alpha) = \{\neg x_i : 1 \leq i \leq m, \alpha \models \neg x_i\}$

   $\mathrm{P}(\alpha) := \{x_i : 1 \leq i \leq m, \alpha \models x_i\}$ linear time (unit propagation)

   Open problem: Can we compute $\mathrm{NL}(\alpha)$ in linear time?

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Quantified Boolean formulas with free variables.

1. BF: Boolean Functions
2. BC: Boolean Circuits
3. PL: Propositional logic
4. QCNF: QBF with free variables and kernel in CNF
5. QHorn^b: QBF with CNF kernel where the bound part of a clause is a Horn clause
6. $\exists$Horn^b: QHorn^b with existential prefix
7. $\exists$2-Horn^b
8. $\exists$ ps-graph$^+$

1. Every quantified Boolean formula is equivalent to a propositional formula.
2. There is no polynomial p
   such that for every n and every Boolean function
   $f : \{0,1\}^n \rightarrow \{0,1\}$ there exists a QBF
   $\Phi(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$ and $|\Phi| \leq p(n)$
   (simple counting argument)

   $|\{\Phi \in QBF : |\Phi| \leq k\}| \leq k^k$
   $\Rightarrow$ ($k = p(n)$ polynomial)

   $|\{\Phi \in QBF : |\Phi| \leq p(n)\}| \leq (p(n)^{p(n)}$
   $BF(n) := |\{f : \{0,1\}^n \rightarrow \{0,1\}| = 2^{2^n}$

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Definition

Let A and B be clasess of formulas. $A =^p B$ iff there exists a polynomial q, such for every $\alpha \in A$ there is an equivalent formula $\beta \in B$ with $|\beta| \leq q(|\alpha|)$ and vice versa.
$A \sqsupset^p B$ iff $\exists$ polynomial q $\forall \beta \in B$ $\exists \alpha \in A : \alpha \approx \beta, |\alpha| \leq q(|\beta|)$.
And $\forall$ polynomials q $\exists \alpha \in A$ $\forall \beta \in B$: If $\alpha \approx \beta$ then $|\beta| > q(|\alpha|)$.

1. $BF(n) \sqsupset^p QBF \sqsupseteq^p PL$
2. $\exists CNF \sqsupseteq^p QHorn^b =^p \exists$ Horn$^b =^p BC \sqsupseteq^p PL$
3. $\exists$2$-$Horn$^b \sqsupseteq^p \exists ps-$graph $=^p PL$

Note: Independent of the running time computing an equivalent formula!

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Existentially quantified CNF with free variables
$\Phi = \exists x_1 \ldots \exists x_n : \phi$ over free variables $Y = y_1, \ldots, y_m$

Definition

$F = (f_1, \ldots, f_n)$ (Boolean functions represented as propositional formulas, $f_i(y_1, \ldots, y_m)$) is an **equivalence model** for Φ
iff
$\Phi \approx \phi[x_1/f_1(Y), \ldots, x_n/f_n(Y)]$

Example: $\Phi = \exists x : (y_1 \vee x) \wedge (\neg x \vee y_2) \approx (y_1 \vee y_2)$

$f_x(y_1, y_2) = \neg y_1$ then

$$\Phi \approx \phi[x/f(y_1, y_2)] \approx (y_1 \vee \neg y_1) \wedge (y_1 \vee y_2)$$

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Problem: ($\Phi = \exists x_1 \ldots \exists x_n : \phi \in \exists$CNF, α propositional formulas)

Size of models versus size of equivalent propositional formulas

Observation: Let F be a model for Φ. Then there is a propositional formula α: $\alpha \approx \Phi$ and $|\alpha| \leq |F| \cdot |\Phi|$.

Open problem: Does there exist a polynomial p, such that for every $\Phi \in \exists$ CNF:
if $\alpha \approx \Phi$ then there exists a model F for Φ with $|F| \leq p(|\alpha|)$?

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Problem: Lower and upper bounds for the size of models.

Reduction to formulas ($\exists$MU$^+$)

$$\Phi = \exists X : \bigwedge_{1 \leq i \leq n} (\varphi_i \vee y_i)$$

φ_i clause over variables X, y_i free variables

$\forall i \exists \alpha \in MU : \alpha \subseteq \{\varphi_1, \ldots, \varphi_n\}$ and $\varphi_i \in \alpha$

(every clause φ_i belongs to a minimal unsatisfiable sub-formula)

Examples:

$$\Phi_1 = \exists x : (x \vee y_1) \wedge (\neg x \vee y_2) \approx (y_1 \vee y_2)$$

MU-subset: $\{x, \neg x\}$

$$\Phi_2 = \exists a \exists b : (a \vee y_1) \wedge (\neg a \vee b \vee y_2) \wedge (\neg a \vee y_3) \wedge (\neg b \vee y_4)$$

MU-subsets: $\{a, \neg a\}, \{a, (\neg a \vee b), \neg b\}$

$$\Phi_2 \approx (y_1 \vee y_3) \wedge (y_1 \vee y_2 \vee y_4)$$

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Notation

$\Phi \in \exists\mathrm{MU}^+$, $\Phi = \exists X : \bigwedge_{1 \leq i \leq n}(\varphi_i \vee y_i)$, and $\varphi = \{\varphi_1, \ldots, \varphi_n\}$

$S(\varphi) := \{\alpha \subseteq \varphi : \alpha \in MU\}$

For $\alpha \in S(\varphi)$: $Y(\alpha) := \{y_i : \varphi_i \in \alpha\}$

Observation:
$\Phi = \exists X : \bigwedge_{1 \leq i \leq n}(\varphi_i \vee y_i) \approx \bigwedge_{\alpha \in S(\varphi)} Y(\alpha)$

Single MU

$\Phi = \exists X : \bigwedge_{1 \leq i \leq m}(\varphi_i \vee y_i) \approx (y_1 \vee \ldots \vee y_m)$ and $\varphi \in$ MU ($S(\varphi) = \{\varphi\}$)

Construct a model $F = (f_{x_1}, \ldots, f_{x_n})$ as follows:

Since φ is minimal unsatisfiable, for every clause φ_j there is a truth assignment v_j satisfying $\varphi \setminus \varphi_j$.

For j they might be several satisfying truth assignments v_j. We choose an arbitrary, but fixed v_j.

We define for every variable $x_i (1 \leq i \leq n)$ a Boolean function $f_{x_i}(y_1, \ldots, y_m)$ represented as propositional DNF-formula as follows:

$f_{x_i}(y_1, \ldots, y_m) := \bigvee_{1 \leq j \leq m, v_j(x_i)=1} (\neg y_1 \wedge \ldots \wedge \neg y_{j-1} \wedge y_j)$

Then $F = (f_{x_1}, \ldots, f_{x_n})$ is a model for Φ. $(|F| \leq m^3)$

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{Upper bound}

$\Phi = \exists X : \bigwedge_{1 \leq i \leq m}(\varphi_i \vee y_i) \in \exists \mathrm{MU}^+$, $\varphi := \bigwedge_{1 \leq i \leq m} \varphi_i$

Theorem

(k minimal unsatisfiable sub-formulas)
If φ contains at most k MU-subformulas,
1. then Φ has a model of size $\leq km^{k+2}$.
2. then there is a propositionel formula $\alpha \approx \Phi$ with $|\alpha| \leq km$

Open Problem: Gap between length of models and equivalent formulas?

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{Upper bound}

$\Phi = \exists X : \bigwedge_{1 \leq i \leq m} (\varphi_i \vee y_i) \in \exists \mathrm{MU}^+$, $\varphi := \bigwedge_{1 \leq i \leq m} \varphi_i$

Theorem

(k minimal unsatisfiable sub-formulas)
If φ contains at most k MU-subformulas,
1. then Φ has a model of size $\leq km^{k+2}$.
2. then there is a propositionel formula $\alpha \approx \Phi$ with $|\alpha| \leq km$

Open Problem: Gap between length of models and length of equivalent formulas?

Lower Bounds: Single MU with **deficiency 1**:
$\Phi = \exists X : \bigwedge_{1 \leq i \leq m}(\varphi_i \vee y_i) \in \exists\text{MU}^+$, $\varphi := \bigwedge_{1 \leq i \leq m} \varphi_i$ in MU(1)

Theorem

1. *upper bound* m^3
2. *if* φ *is marginal then a lower bound is* $\frac{(m-1)^2}{4} + \frac{m-1}{2}$ *(few satisfying truth assignments)*
3. *if* φ *is in MAX-MU then a lower bound is* $\frac{m}{2} \cdot log_2(m)$ *(max. number of satisfying truth assignments)*

1. (improve upper bound)
2. lower bounds for single MU with deficiency k

Borderline: minimal unsatisfiable Horn formulas have deficiency 1.
$\exists \text{Horn}^b =^p \text{BC}$.

Model size for $\exists(\text{2-Horn} \cap \text{MU})^+$

Case 1: $\exists\text{ps-graph}^+ =^P$ propositional formulas

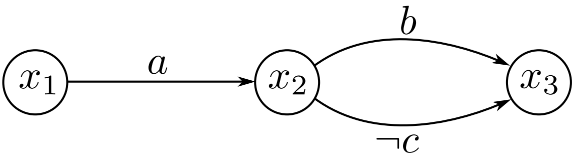

Figure: ps-graph

$$\exists x_1 \exists x_2 \exists x_3 : x_1, (\neg x_1 \vee x_2 \vee a), (\neg x_2 \vee x_3 \vee b), (\neg x_2 \vee x_3 \vee \neg c), (\neg x_3)$$
$$\approx (a \vee (b \wedge \neg c))$$

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$\exists \text{Horn}^b =^p \text{BC}$.

Model size for $\exists(\text{2-Horn} \cap \text{MU})^+$

Case 1: $\exists \text{ps-graph}^+ =^P$ propositional formulas

Theorem

1. $\exists \text{ps-graph}^+ =^p PL$
2 Formulas in $\exists \text{ps-graph}^+$ have poly-size models.

Borderline: minimal unsatisfiable Horn formulas have deficiency 1.
$\exists\text{Horn}^b =^p$ BC.

Model size for $\exists$(2-Horn $\cap$ MU)$^+$

Case 2: $\exists$DAG(1)$^+$ $\sqsupseteq^p$ propositional formluas ($\sqsupset^p$ or $=^p$ open)

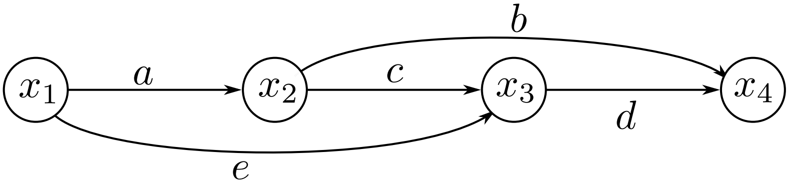

Figure: DAG(1)

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 : x_1, \text{labeled edges}, \neg x_4$$
$$\approx (a \vee c \vee d), (e \vee d), (a \vee b)$$

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Borderline: minimal unsatisfiable Horn formulas have deficiency 1.
$\exists\text{Horn}^b =^p \text{BC}$.

Model size for $\exists(\text{2-Horn} \cap \text{MU})^+$

Case 2: $\exists\text{DAG}(1)^+ \sqsupseteq^p$ propositional formluas ($\sqsupset^p$ or $=^p$ open)

Theorem

Formulas in $\exists DAG(1)^+$ have poly-size models iff
the formulas have poly-size equivalent propositional formulas iff
$\exists DAG(1)^+ =^p PL$

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Summary

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  1. Equivalence Models and Propositional Formulas
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  3. Model size and Deficiency

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Thank You for Your Attention!

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