First-Order Theorem Proving and Vampire

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Outline

First-Order Logic: Exercises

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8. Having proofs is good.
9. Vampire is a first-order theorem prover.

Future and Our Motivation

1. Theorem proving will remain central in software verification and program analysis. The role of theorem proving in these areas will be growing.
2. Theorem provers will be used by a large number of users who do not understand theorem proving and by users with very elementary knowledge of logic.
3. Reasoning with both quantifiers and theories will remain the main challenge in practical applications of theorem proving (at least) for the next decade.
4. Theorem provers will be used in reasoning with very large theories. These theories will appear in knowledge mining and natural language processing.

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More formally: in a group "assuming that $x^2 = 1$ for all x prove that $x \cdot y = y \cdot x$ holds for all x, y."
What is implicit: axioms of the group theory.

$$\forall x(1 \cdot x = x)$$
$$\forall x(x^{-1} \cdot x = 1)$$
$$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

Formulation in First-Order Logic

Axioms (of group theory):	$\forall x(1 \cdot x = x)$
	$\forall x(x^{-1} \cdot x = 1)$
	$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$
Assumptions:	$\forall x(x \cdot x = 1)$
Conjecture:	$\forall x \forall y(x \cdot y = y \cdot x)$

In the TPTP Syntax

The TPTP library (Thousands of Problems for Theorem Provers), `http://www.tptp.org` contains a large collection of first-order problems. For representing these problems it uses the TPTP syntax, which is understood by all modern theorem provers, including Vampire.

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In the TPTP syntax this group theory problem can be written down as follows:

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    ! [X] : mult(e,X) = X).
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Running Vampire of a TPTP file

is easy: simply use

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One can also run Vampire with various options, some of them will be explained later. For example, save the group theory problem in a file `group.tptp` and try

```
vampire --thanks ReRiSE group.tptp
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FOL	TPTP
$\bot, \top$	`$false, $true`
$\neg F$	`~F`
$F_1 \wedge \ldots \wedge F_n$	`F1 & ... & Fn`
$F_1 \vee \ldots \vee F_n$	`F1 | ... | Fn`
$F_1 \rightarrow F_n$	`F1 => Fn`
$(\forall x_1) \ldots (\forall x_n) F$	`! [X1,...,Xn] : F`
$(\exists x_1) \ldots (\exists x_n) F$	`? [X1,...,Xn] : F`

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Proof by Vampire (Slightliy Modified)

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Statistics

```
Version: Vampire 3 (revision 2038)
Termination reason: Refutation

Active clauses: 14
Passive clauses: 28
Generated clauses: 124
Final active clauses: 8
Final passive clauses: 6
Input formulas: 5
Initial clauses: 6

Splitted inequalities: 1

Fw subsumption resolutions: 1
Fw demodulations: 32
Bw demodulations: 12

Forward subsumptions: 53
Backward subsumptions: 1
Fw demodulations to eq. taut.: 6
Bw demodulations to eq. taut.: 1

Forward superposition: 41
Backward superposition: 28
Self superposition: 4

Memory used [KB]: 255
Time elapsed: 0.005 s
```

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- Completely automatic: once you started a proof attempt, it can only be interrupted by terminating the process.

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- Completely automatic: once you started a proof attempt, it can only be interrupted by terminating the process.
- Champion of the CASC world-cup in first-order theorem proving: won CASC 28 times.

Main applications

- Software and hardware verification;
- Static analysis of programs;
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- Program synthesis;
- Writing papers and giving talks at various conferences and schools . . .

What an Automatic Theorem Prover is Expected to Do

Input:

- a set of axioms (first order formulas) or clauses;
- a conjecture (first-order formula or set of clauses).

Output:

- proof (hopefully).

Proof by Refutation

Given a problem with axioms and assumptions $F_1, \ldots, F_n$ and conjecture G,

1. negate the conjecture;
2. establish unsatisfiability of the set of formulas $F_1, \ldots, F_n, \neg G$.

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In this formulation the negation of the conjecture $\neg G$ is treated like any other formula. In fact, Vampire (and other provers) internally treat conjectures differently, to make proof search more goal-oriented.

General Scheme (simplified)

- Read a problem;
- Determine proof-search options to be used for this problem;
- Preprocess the problem;
- Convert it into CNF;
- Run a saturation algorithm on it, try to derive $\bot$.
- If $\bot$ is derived, report the result, maybe including a refutation.

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Trying to derive $\bot$ using a saturation algorithm is the hardest part, which in practice may not terminate or run out of memory.

Outline

Inference System

- inference has the form

$$\frac{F_1 \quad \ldots \quad F_n}{G} \ ,$$

  where $n \geq 0$ and $F_1, \ldots, F_n, G$ are formulas.
- The formula G is called the conclusion of the inference;
- The formulas $F_1, \ldots, F_n$ are called its premises.
- An inference rule R is a set of inferences.
- Every inference $I \in R$ is called an instance of R.
- An Inference system $\mathbb{I}$ is a set of inference rules.
- Axiom: inference rule with no premises.

Inference System: Example

Represent the natural number n by the string $\underbrace{|\ldots|}_{n \text{ times}}\varepsilon$.

The following inference system contains 6 inference rules for deriving equalities between expressions containing natural numbers, addition $+$ and multiplication $\cdot$.

$$\frac{}{\varepsilon = \varepsilon}\ (\varepsilon) \qquad \frac{x = y}{|x = |y}\ (|)$$

$$\frac{}{\varepsilon + x = x}\ (+_1) \qquad \frac{x + y = z}{|x + y = |z}\ (+_2)$$

$$\frac{}{\varepsilon \cdot x = \varepsilon}\ (\cdot_1) \qquad \frac{x \cdot y = u \quad y + u = z}{|x \cdot y = z}\ (\cdot_2)$$

Derivation, Proof

- Derivation in an inference system $\mathbb{I}$: a tree built from inferences in $\mathbb{I}$.
- If the root of this derivation is E, then we say it is a derivation of E.
- Proof of E: a finite derivation whose leaves are axioms.
- Derivation of E from $E_1, \ldots, E_m$: a finite derivation of E whose every leaf is either an axiom or one of the expressions $E_1, \ldots, E_m$.

Examples

For example,

$$\frac{||\varepsilon + |\varepsilon = |||\varepsilon}{|||\varepsilon + |\varepsilon = ||||\varepsilon}\ (+_2)$$

is an inference that is an instance (special case) of the inference rule

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The axiom

$$\frac{}{\varepsilon + |||\varepsilon = |||\varepsilon}\ (+_1)$$

is an instance of the rule

$$\frac{}{\varepsilon + x = x}\ (+_1)$$

Proof in this Inference System

Proof of $||\varepsilon \cdot ||\varepsilon = ||||\varepsilon$ (that is, $2 \cdot 2 = 4$).

$$\dfrac{\dfrac{}{\varepsilon \cdot ||\varepsilon = \varepsilon}\,(\cdot_1) \quad \dfrac{\dfrac{\dfrac{}{\varepsilon + \varepsilon = \varepsilon}\,(+_1)}{|\varepsilon + \varepsilon = |\varepsilon}\,(+_2)}{||\varepsilon + \varepsilon = ||\varepsilon}\,(+_2)}{|\varepsilon \cdot ||\varepsilon = ||\varepsilon}\,(\cdot_2) \quad \dfrac{\dfrac{\dfrac{}{\varepsilon + ||\varepsilon = ||\varepsilon}\,(+_1)}{|\varepsilon + ||\varepsilon = |||\varepsilon}\,(+_2)}{||\varepsilon + ||\varepsilon = ||||\varepsilon}\,(+_2)}{||\varepsilon \cdot ||\varepsilon = ||||\varepsilon}\,(\cdot_2).$$

Derivation in this Inference System

Derivation of $||\varepsilon \cdot ||\varepsilon = |||||\varepsilon$ from $\varepsilon + ||\varepsilon = |||\varepsilon$ (that is, $2 + 2 = 5$ from $0 + 2 = 3$).

$$\dfrac{\dfrac{}{\varepsilon \cdot ||\varepsilon = \varepsilon}\,(\cdot_1) \quad \dfrac{\dfrac{\dfrac{}{\varepsilon + \varepsilon = \varepsilon}\,(+_1)}{|\varepsilon + \varepsilon = |\varepsilon}\,(+_2)}{||\varepsilon + \varepsilon = ||\varepsilon}\,(+_2)}{|\varepsilon \cdot ||\varepsilon = ||\varepsilon}\,(\cdot_2) \quad \dfrac{\dfrac{\varepsilon + ||\varepsilon = |||\varepsilon}{|\varepsilon + ||\varepsilon = ||||\varepsilon}\,(+_2)}{||\varepsilon + ||\varepsilon = |||||\varepsilon}\,(+_2)}{||\varepsilon \cdot ||\varepsilon = ||||\varepsilon}\,(\cdot_2).$$

Arbitrary First-Order Formulas

- A first-order signature (vocabulary): function symbols (including constants), predicate symbols. Equality is part of the language.
- A set of variables.
- Terms are built using variables and function symbols. For example, $f(x) + g(x)$.
- Atoms, or atomic formulas are obtained by applying a predicate symbol to a sequence of terms. For example, $p(a, x)$ or $f(x) + g(x) \geq 2$.
- Formulas: built from atoms using logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\forall$, $\exists$. For example, $(\forall x)x = 0 \vee (\exists y)y > x$.

Clauses

- Literal: either an atom A or its negation $\neg A$.
- Clause: a disjunction $L_1 \vee \ldots \vee L_n$ of literals, where $n \geq 0$.

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- A formula in Clausal Normal Form (CNF): a conjunction of clauses.
- A clause is ground if it contains no variables.
- If a clause contains variables, we assume that it implicitly universally quantified. That is, we treat $p(x) \vee q(x)$ as $\forall x(p(x) \vee q(x))$.

Binary Resolution Inference System

The binary resolution inference system, denoted by $\mathbb{BR}$ is an inference system on propositional clauses (or ground clauses). It consists of two inference rules:

- Binary resolution, denoted by BR:

$$\frac{p \vee C_1 \quad \neg p \vee C_2}{C_1 \vee C_2} \text{ (BR).}$$

- Factoring, denoted by Fact:

$$\frac{L \vee L \vee C}{L \vee C} \text{ (Fact).}$$

Soundness

- An inference is sound if the conclusion of this inference is a logical consequence of its premises.
- An inference system is sound if every inference rule in this system is sound.

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$\mathbb{BR}$ is sound.

Consequence of soundness: let S be a set of clauses. If $\square$ can be derived from S in $\mathbb{BR}$, then S is unsatisfiable.

Example

Consider the following set of clauses

$$\{\neg p \vee \neg q,\ \neg p \vee q,\ p \vee \neg q,\ p \vee q\}.$$

The following derivation derives the empty clause from this set:

$$\dfrac{\dfrac{p \vee q \quad p \vee \neg q}{p \vee p}\,(\text{BR})}{p}\,(\text{Fact}) \qquad \dfrac{\dfrac{\neg p \vee q \quad \neg p \vee \neg q}{\neg p \vee \neg p}\,(\text{BR})}{\neg p}\,(\text{Fact})$$
$$\dfrac{}{\square}\,(\text{BR})$$

Hence, this set of clauses is unsatisfiable.

Can this be used for checking (un)satisfiability

1. What happens when the empty clause cannot be derived from S?
2. How can one search for possible derivations of the empty clause?

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1. Completeness.

   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.

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   Let S be an unsatisfiable set of clauses. Then there exists a derivation of $\square$ from S in $\mathbb{BR}$.

2. We have to formalize search for derivations.

However, before doing this we will introduce a slightly more refined inference system.

Selection Function

A literal selection function selects literals in a clause.

- If C is non-empty, then at least one literal is selected in C.

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Note: selection function does not have to be a function. It can be any oracle that selects literals.

Binary Resolution with Selection

We introduce a family of inference systems, parametrised by a literal selection function σ.
The binary resolution inference system, denoted by $\mathbb{BR}_\sigma$, consists of two inference rules:

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$$\frac{\underline{p} \vee C_1 \quad \underline{\neg p} \vee C_2}{C_1 \vee C_2} \text{ (BR).}$$

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- Positive factoring, denoted by Fact:

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact).}$$

Completeness?

Binary resolution with selection may be incomplete, even when factoring is unrestricted (also applied to negative literals).

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Consider this set of clauses:

(1) $\neg q \vee \underline{r}$
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(3) $\neg r \vee \underline{\neg q}$
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It is unsatisfiable:

(8)	$q \vee p$	(6, 7)
(9)	q	(2, 8)
(10)	r	(1, 9)
(11)	$\neg q$	(3, 10)
(12)	$\square$	(9, 11)

Note the linear representation of derivations (used by Vampire and many other provers).

However, any inference with selection applied to this set of clauses give either a clause in this set, or a clause containing a clause in this set.

Literal Orderings

Take any well-founded ordering $\succ$ on atoms, that is, an ordering such that there is no infinite decreasing chain of atoms:

$$A_0 \succ A_1 \succ A_2 \succ \cdots$$

In the sequel $\succ$ will always denote a well-founded ordering.

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- If $p \succ q$, then $p \succ \neg q$ and $\neg p \succ q$;
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- $\neg p \succ p$.

Exercise: prove that the induced ordering on literals is well-founded too.

Orderings and Well-Behaved Selections

Fix an ordering $\succ$. A literal selection function is well-behaved if

- If all selected literals are positive, then all maximal (w.r.t. $\succ$) literals in C are selected.

In other words, either a negative literal is selected, or all maximal literals must be selected.

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In other words, either a negative literal is selected, or all maximal literals must be selected.

To be well-behaved, we sometimes must select more than one different literal in a clause. Example: $p \vee p$ or $p(x) \vee p(y)$.

Completeness of Binary Resolution with Selection

Binary resolution with selection is complete for every well-behaved selection function.

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Consider our previous example:

(1) $\neg q \vee \underline{r}$
(2) $\neg p \vee \underline{q}$
(3) $\neg r \vee \underline{\neg q}$
(4) $\neg q \vee \underline{\neg p}$
(5) $\neg p \vee \underline{\neg r}$
(6) $\neg r \vee \underline{p}$
(7) $r \vee q \vee \underline{p}$

A well-behave selection function must satisfy:

1. $r \succ q$, because of (1)
2. $q \succ p$, because of (2)
3. $p \succ r$, because of (6)

There is no ordering that satisfies these conditions.

End of Lecture 1

Slides for lecture 1 ended here . . .

Outline

How to Establish Unsatisfiability?

Completess is formulated in terms of derivability of the empty clause $\square$ from a set S_0 of clauses in an inference system $\mathbb{I}$. However, this formulations gives no hint on how to search for such a derivation.

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Idea:

- Take a set of clauses S (the search space), initially $S = S_0$. Repeatedly apply inferences in $\mathbb{I}$ to clauses in S and add their conclusions to S, unless these conclusions are already in S.
- If, at any stage, we obtain $\square$, we terminate and report unsatisfiability of S_0.

How to Establish Satisfiability?

When can we report satisfiability?

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In first-order logic it is often the case that all saturated sets are infinite (due to undecidability), so in practice we can never build a saturated set.

The process of trying to build one is referred to as saturation.

Saturated Set of Clauses

Let $\mathbb{I}$ be an inference system on formulas and S be a set of formulas.

- S is called saturated with respect to $\mathbb{I}$, or simply $\mathbb{I}$-saturated, if for every inference of $\mathbb{I}$ with premises in S, the conclusion of this inference also belongs to S.
- The closure of S with respect to $\mathbb{I}$, or simply $\mathbb{I}$-closure, is the smallest set S' containing S and saturated with respect to $\mathbb{I}$.

Inference Process

Inference process: sequence of sets of formulas $S_0, S_1, \ldots$, denoted by

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

$(S_i \Rightarrow S_{i+1})$ is a step of this process.

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We say that this step is an $\mathbb{I}$-step if

1. there exists an inference

$$\frac{F_1 \quad \ldots \quad F_n}{F}$$

   in $\mathbb{I}$ such that $\{F_1, \ldots, F_n\} \subseteq S_i$;
2. $S_{i+1} = S_i \cup \{F\}$.

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2. $S_{i+1} = S_i \cup \{F\}$.

An $\mathbb{I}$-inference process is an inference process whose every step is an $\mathbb{I}$-step.

Property

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an $\mathbb{I}$-inference process and a formula F belongs to some S_i. Then S_i is derivable in $\mathbb{I}$ from S_0. In particular, every S_i is a subset of the $\mathbb{I}$-closure of S_0.

Limit of a Process

The limit of an inference process $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ is the set of formulas $\bigcup_i S_i$.

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In other words, the limit is the set of all derived formulas.

Suppose that we have an infinite inference process such that S_0 is unsatisfiable and we use a sound and complete inference system.

Question: does completeness imply that the limit of the process contains the empty clause?

Fairness

Let $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be an inference process with the limit S_∞. The process is called fair if for every $\mathbb{I}$-inference

$$\frac{F_1 \quad \ldots \quad F_n}{F} \ ,$$

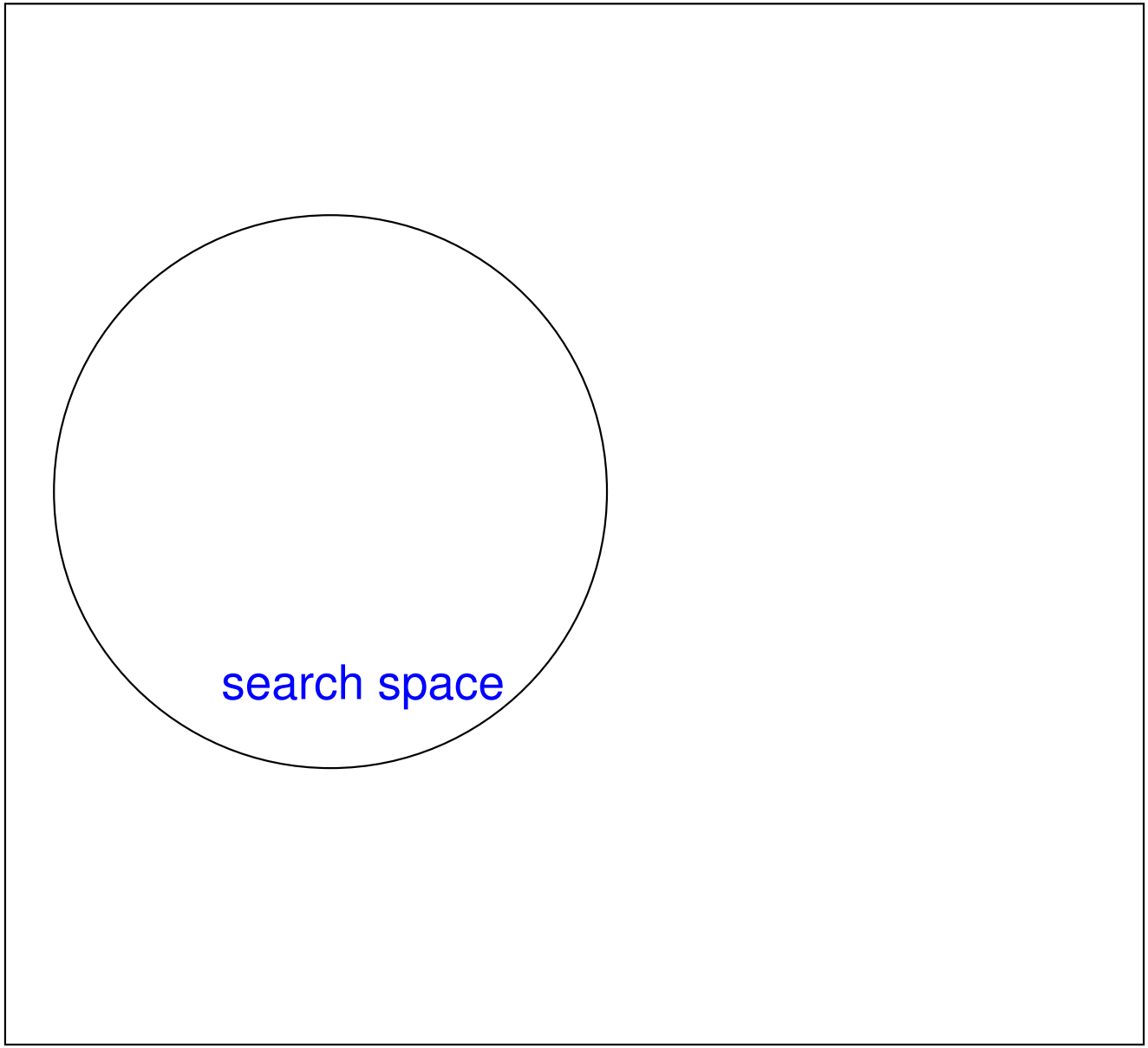
if $\{F_1, \ldots, F_n\} \subseteq S_\infty$, then there exists i such that $F \in S_i$.

Completeness, reformulated

Theorem Let $\mathbb{I}$ be an inference system. The following conditions are equivalent.

1. $\mathbb{I}$ is complete.
2. For every unsatisfiable set of formulas S_0 and any fair $\mathbb{I}$-inference process with the initial set S_0, the limit of this inference process contains $\square$.

Fair Saturation Algorithms: Inference Selection by Clause Selection



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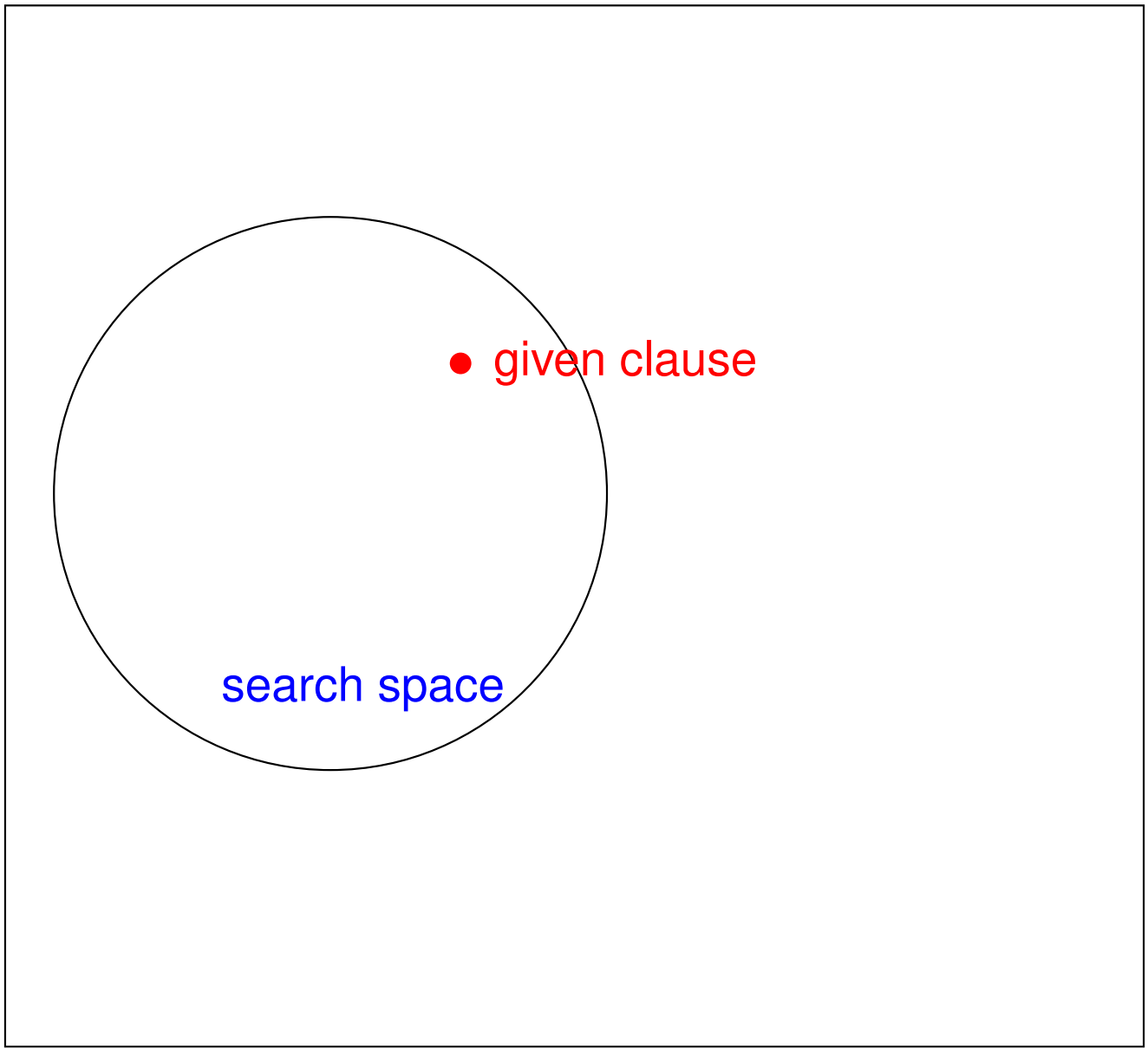

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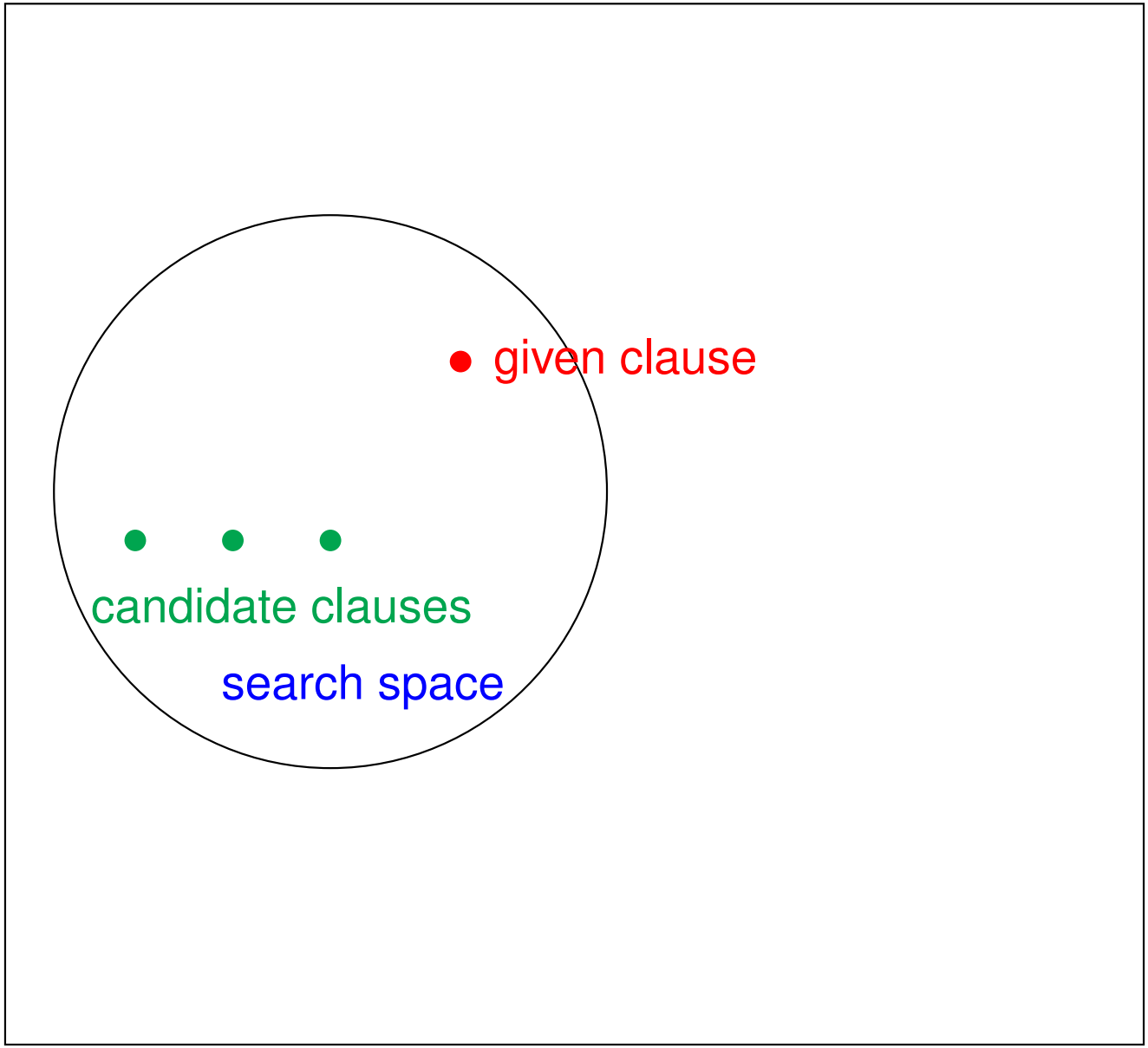

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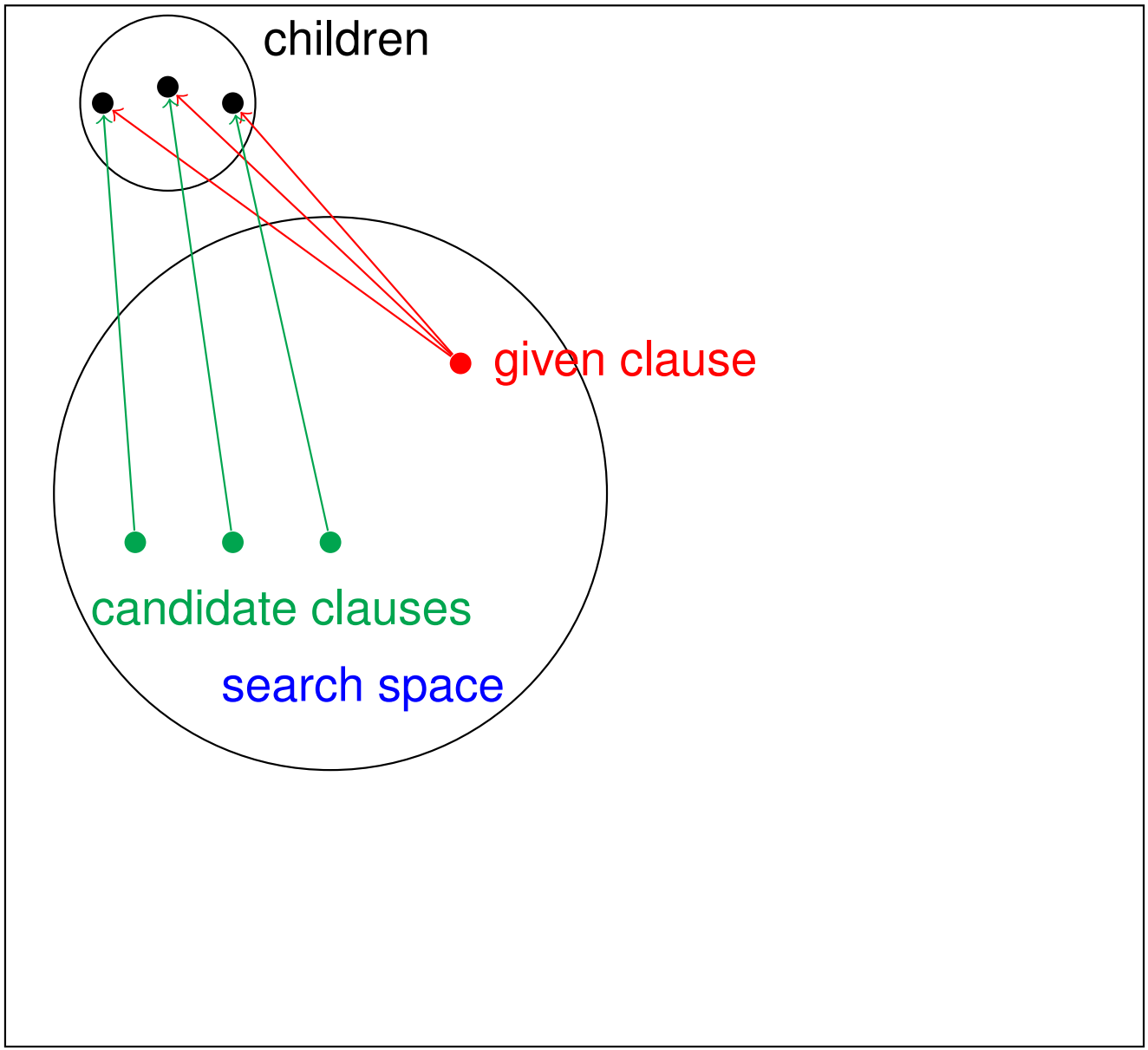

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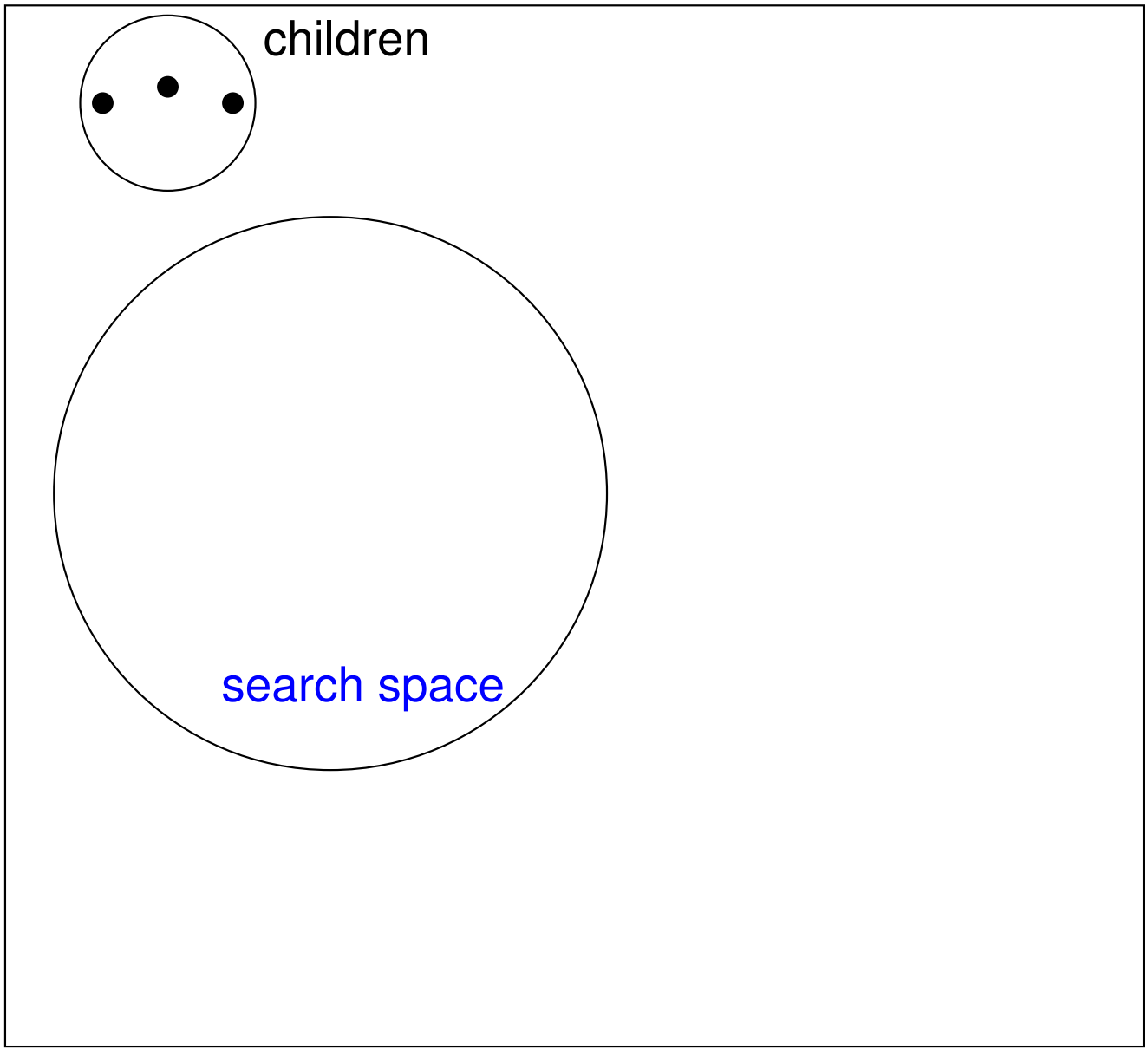

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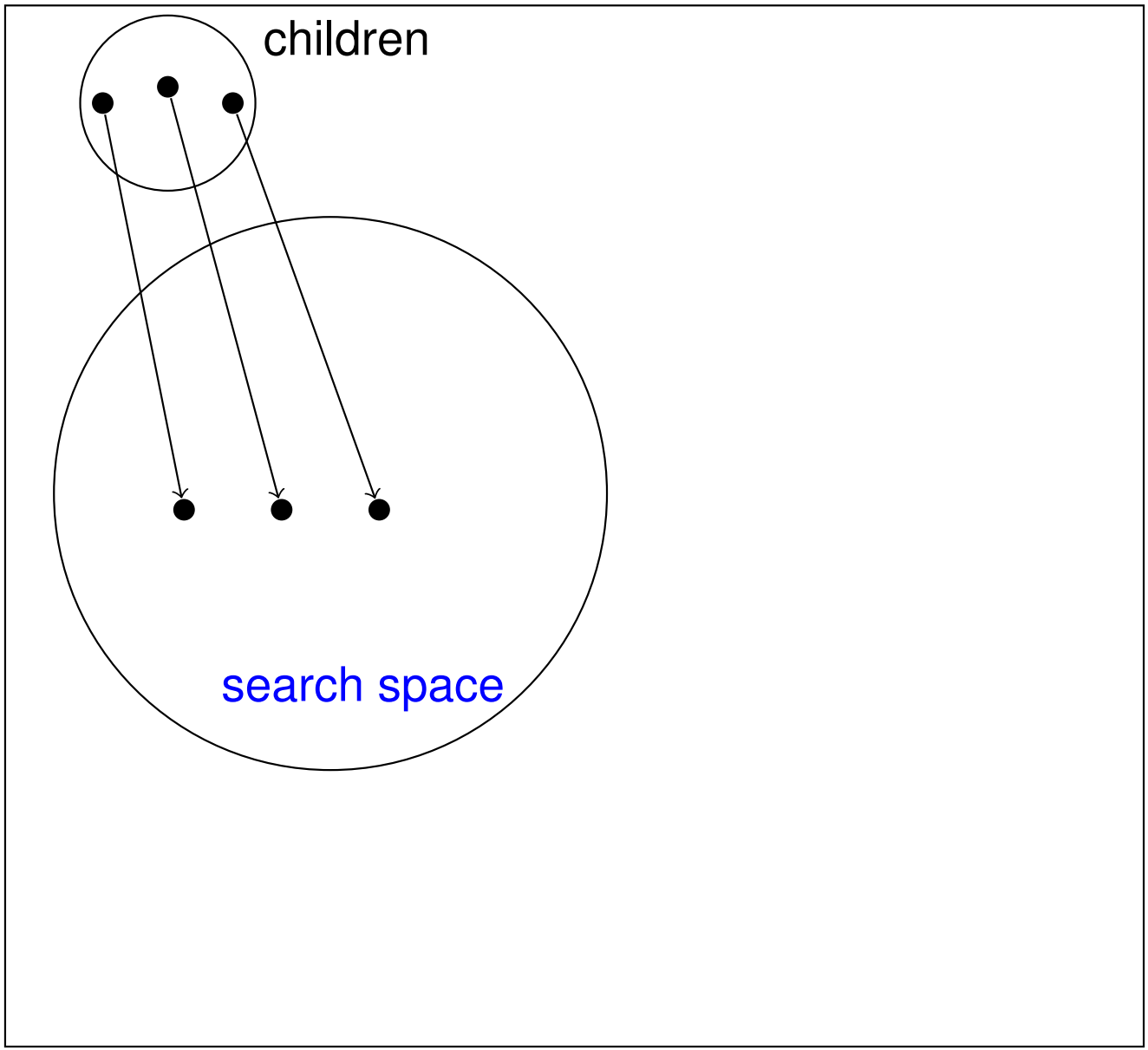

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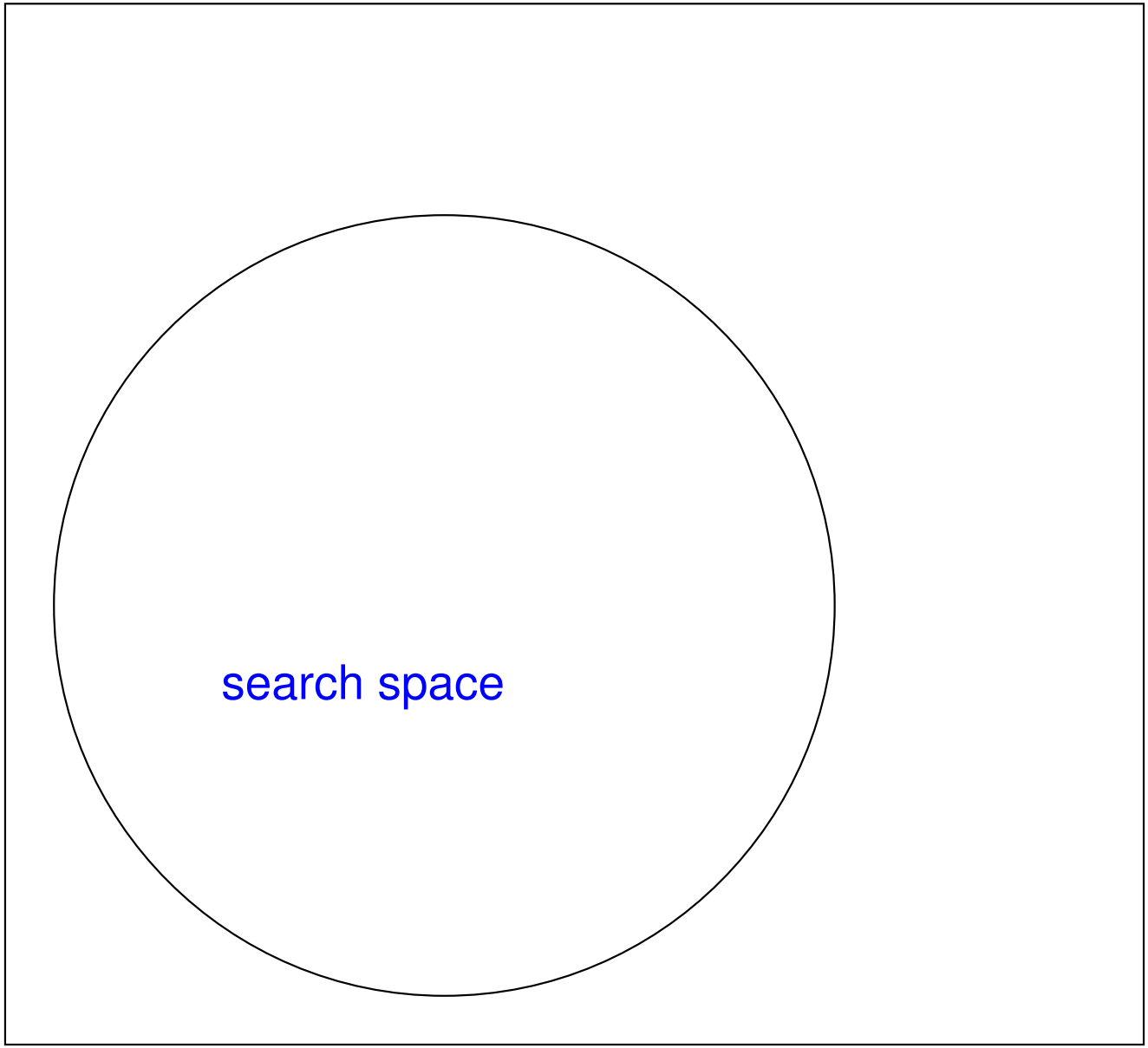

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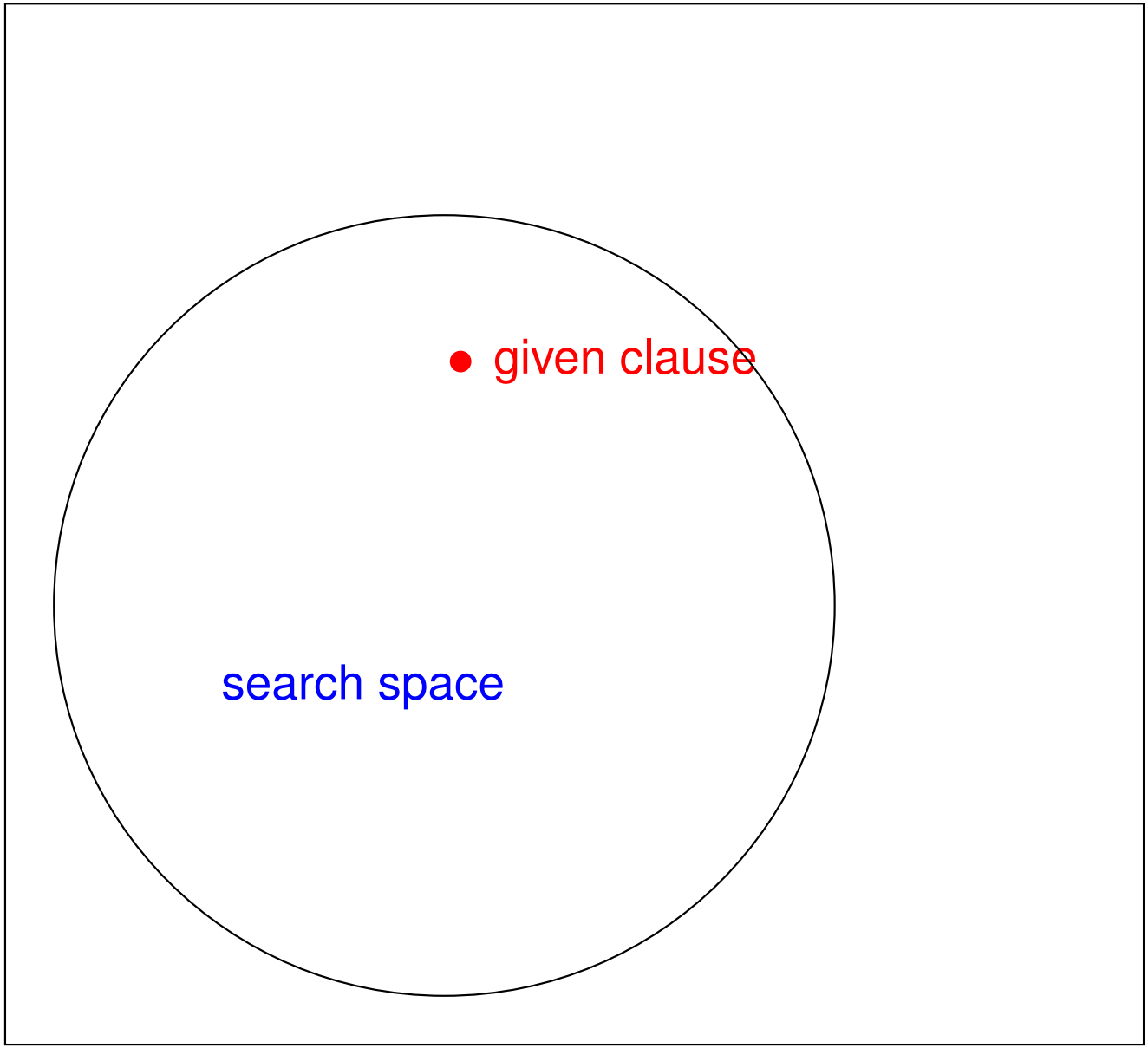

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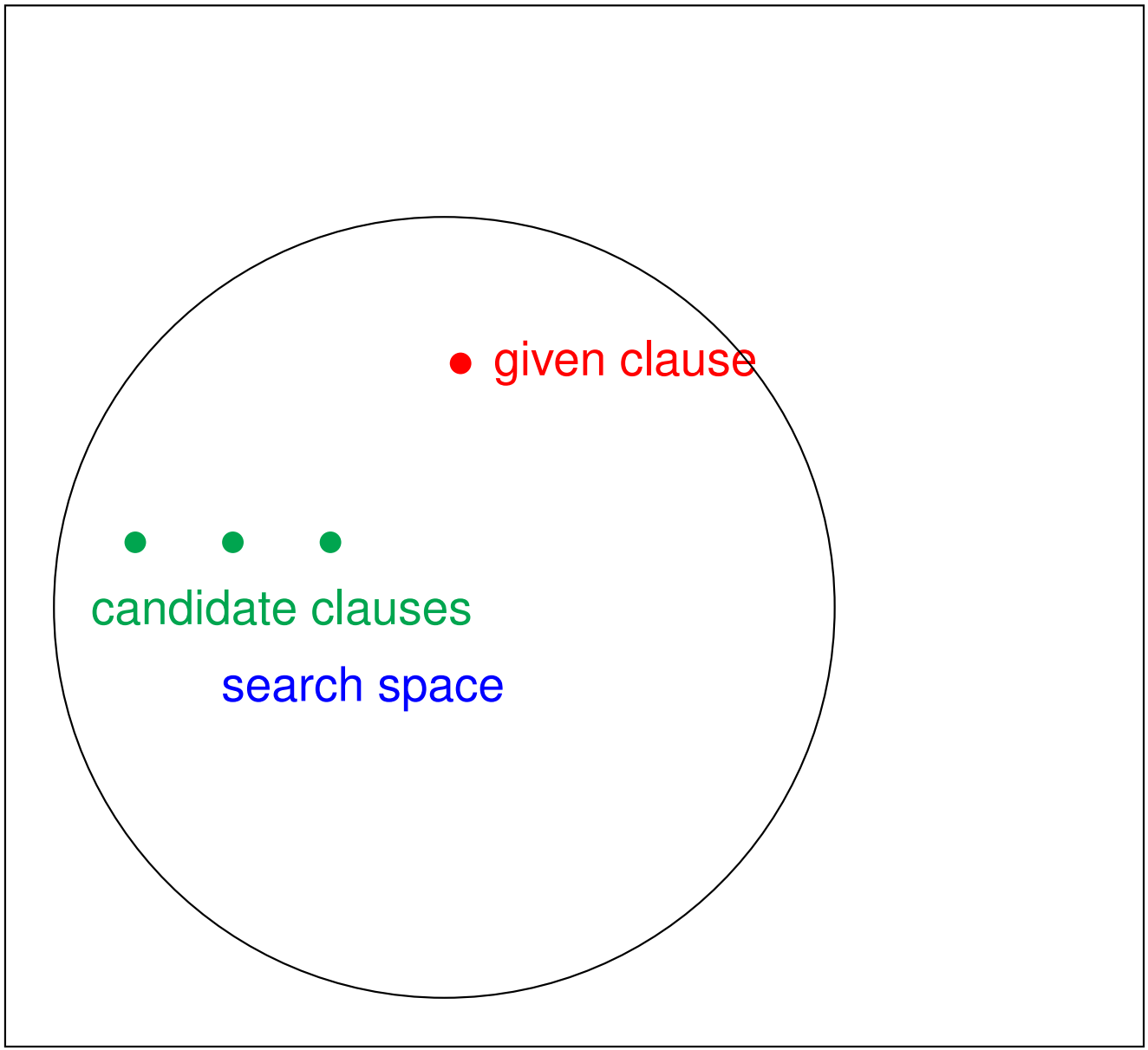

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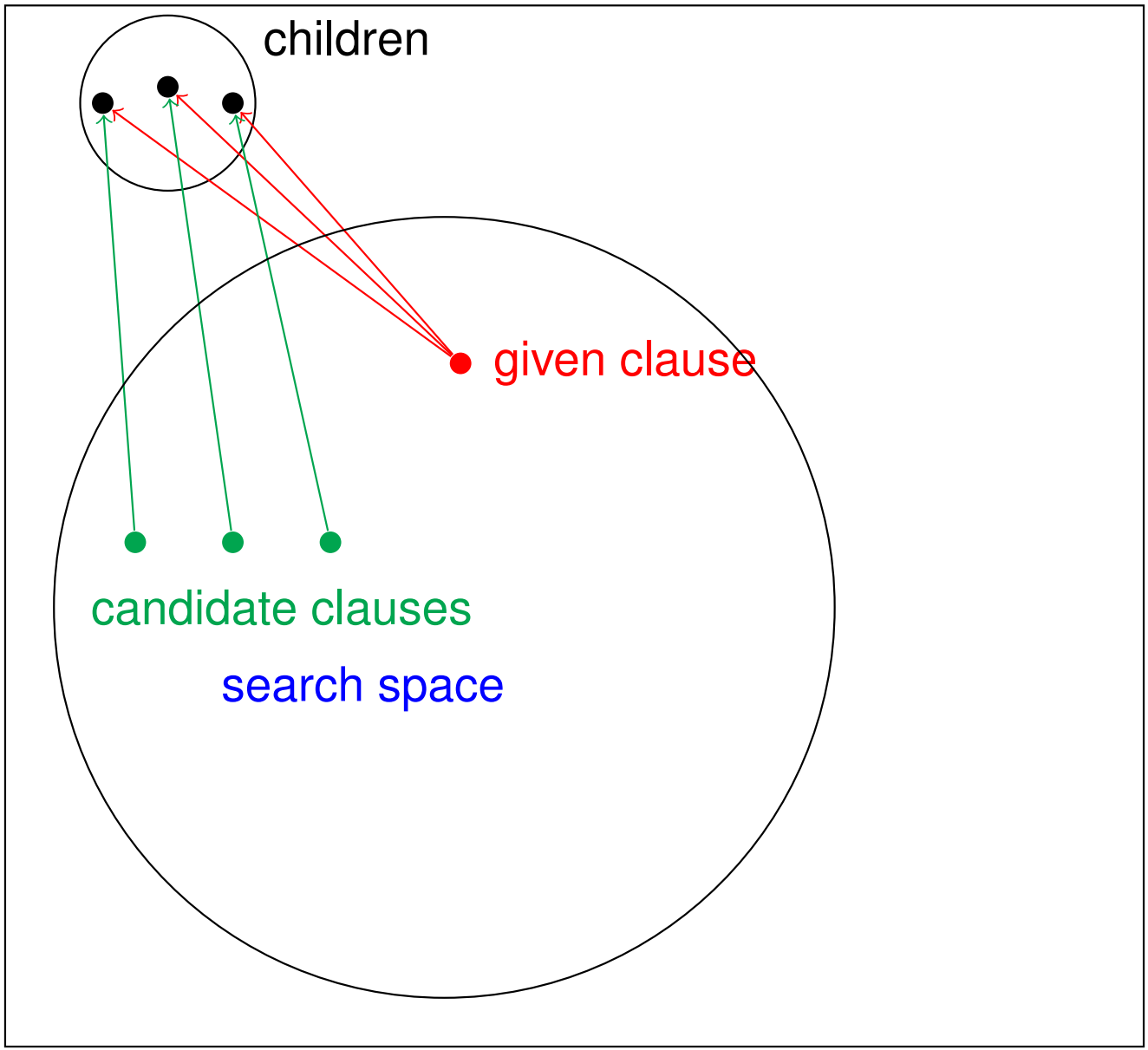

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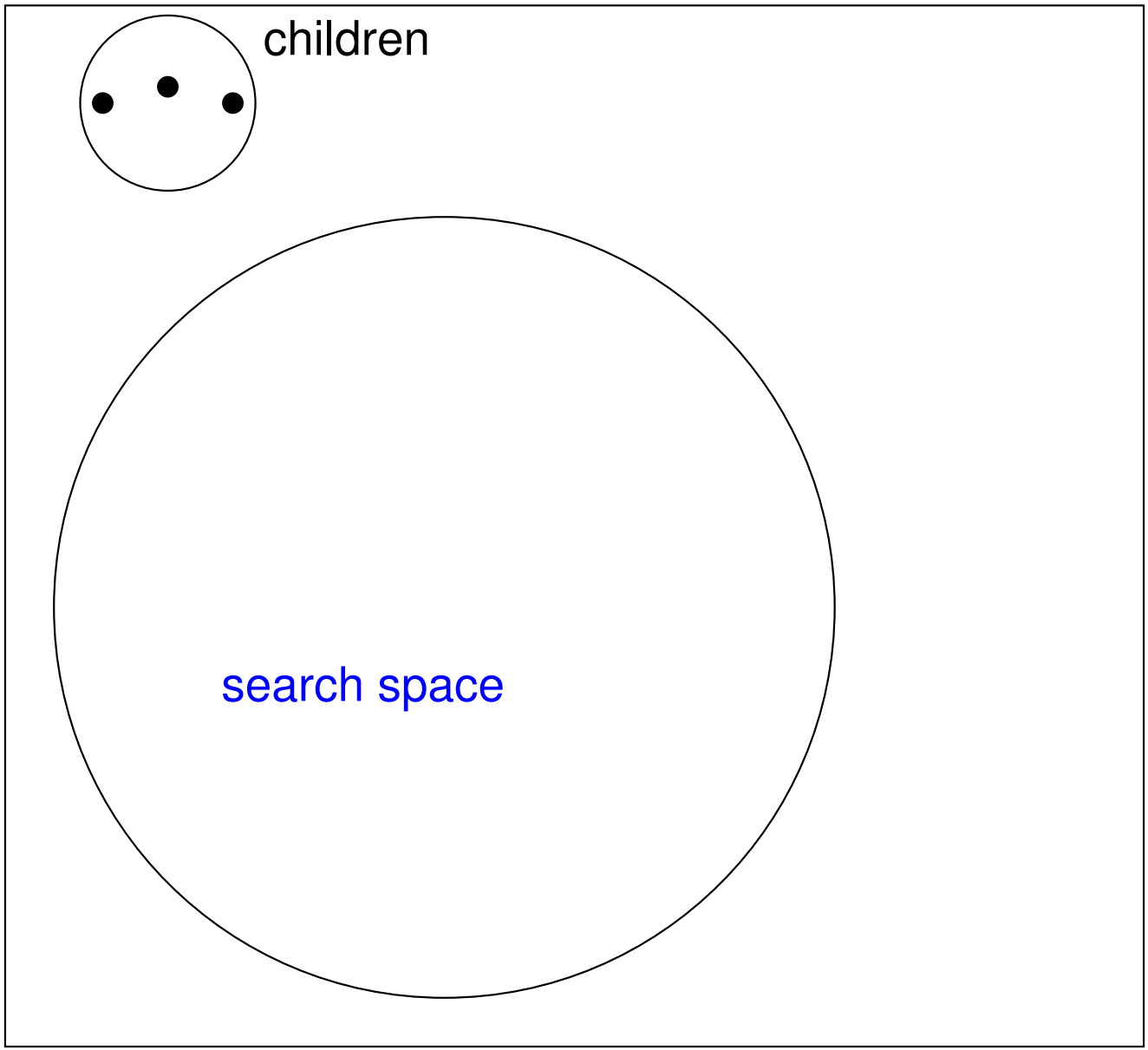

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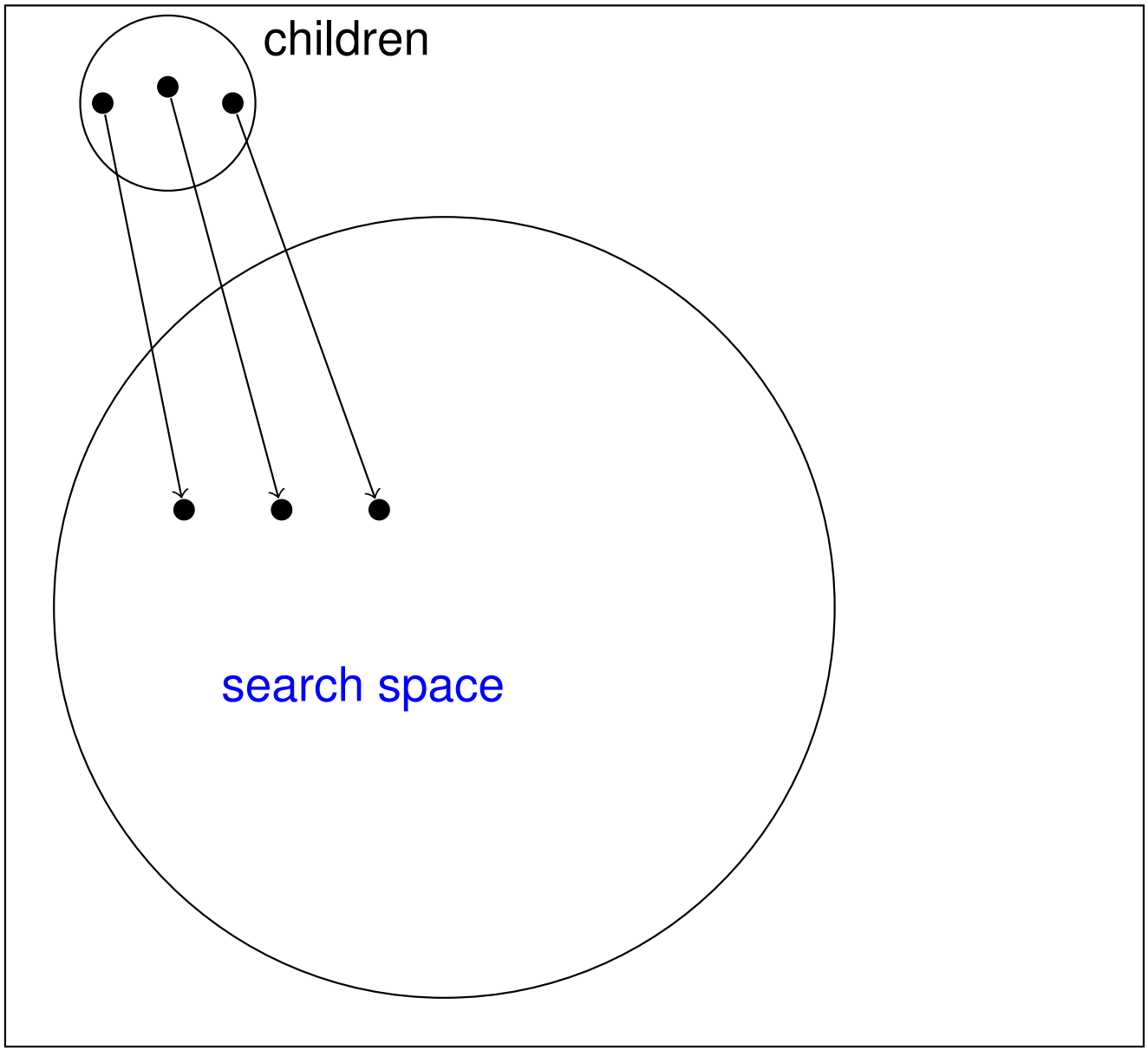

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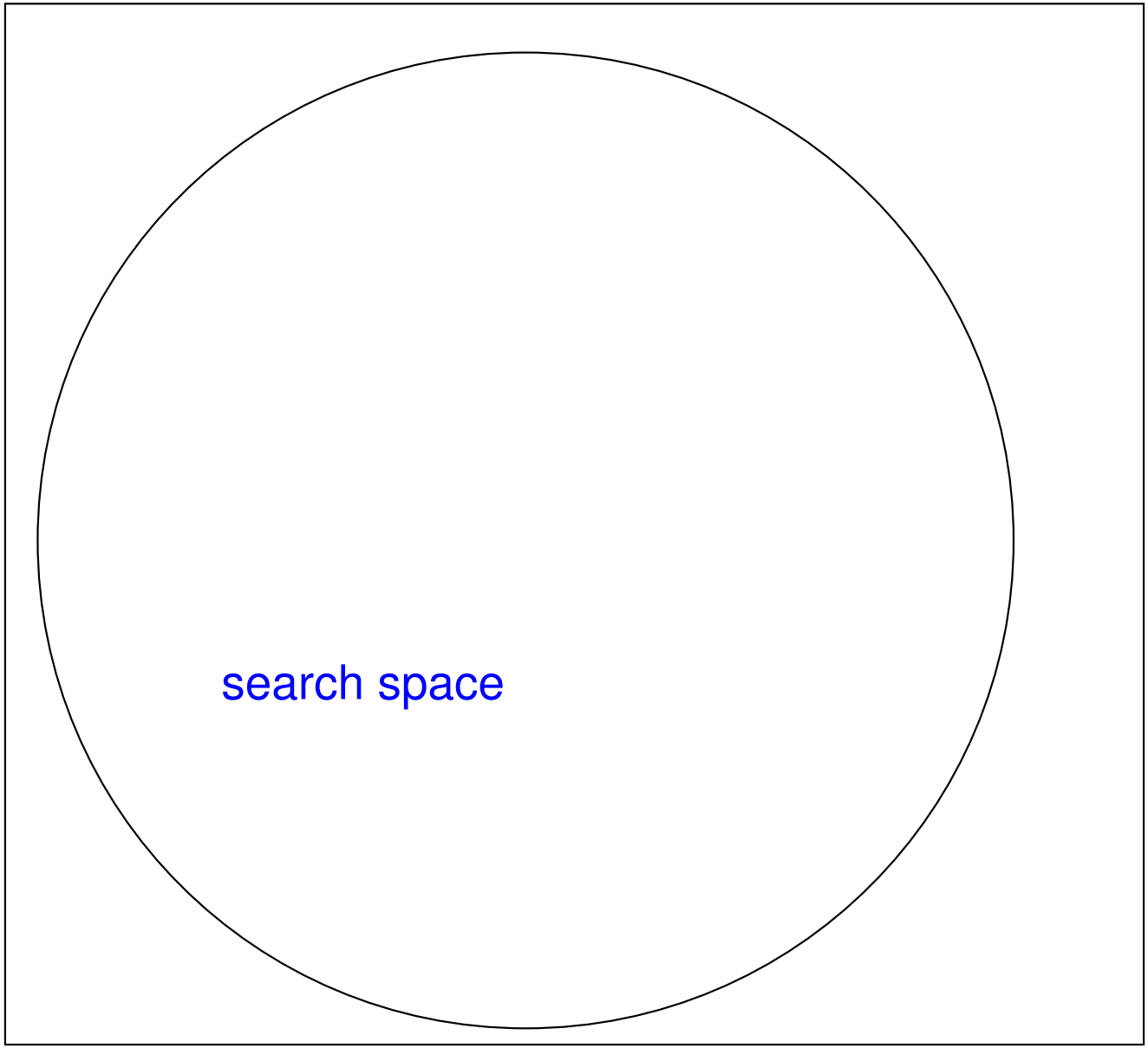

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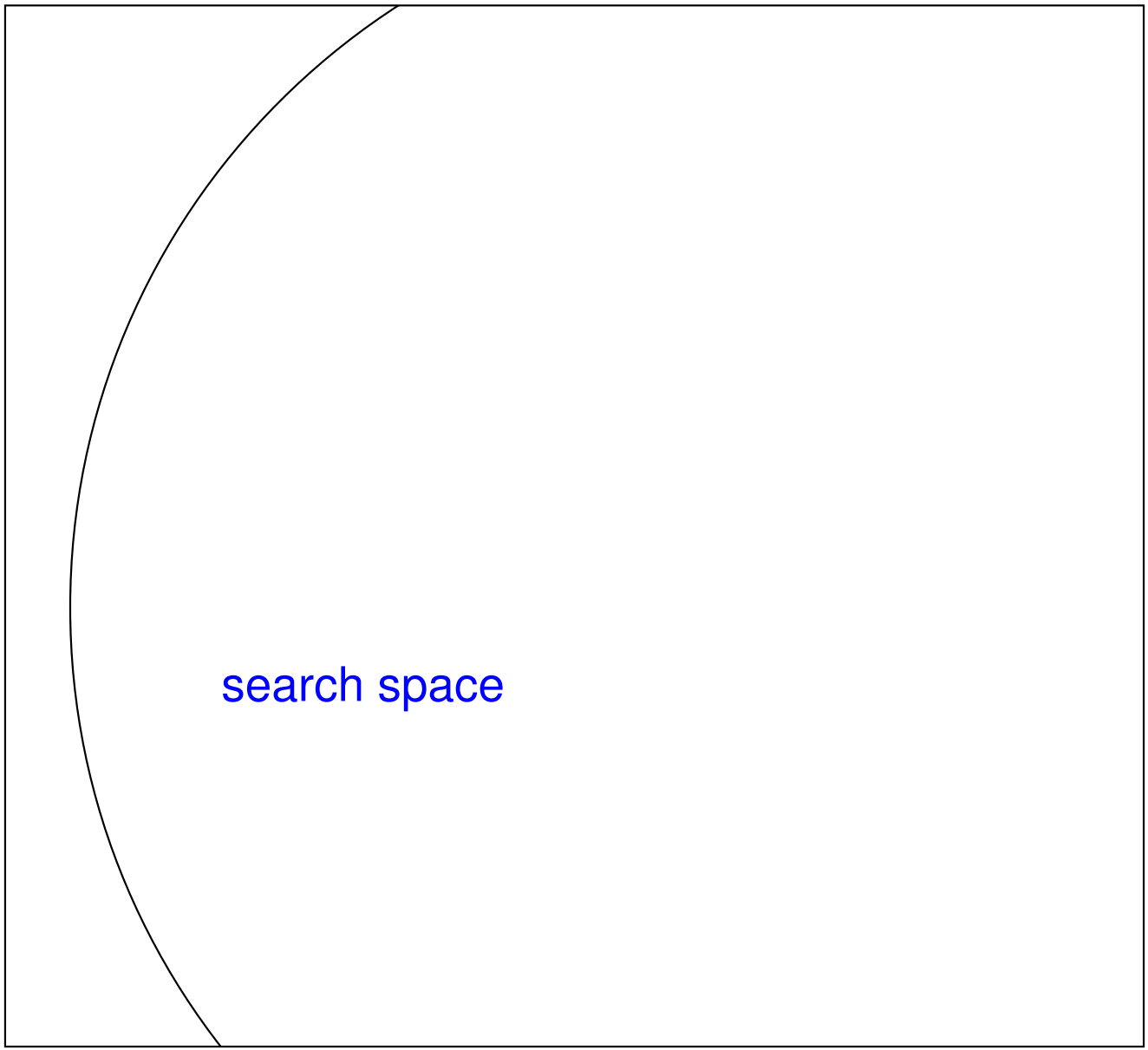

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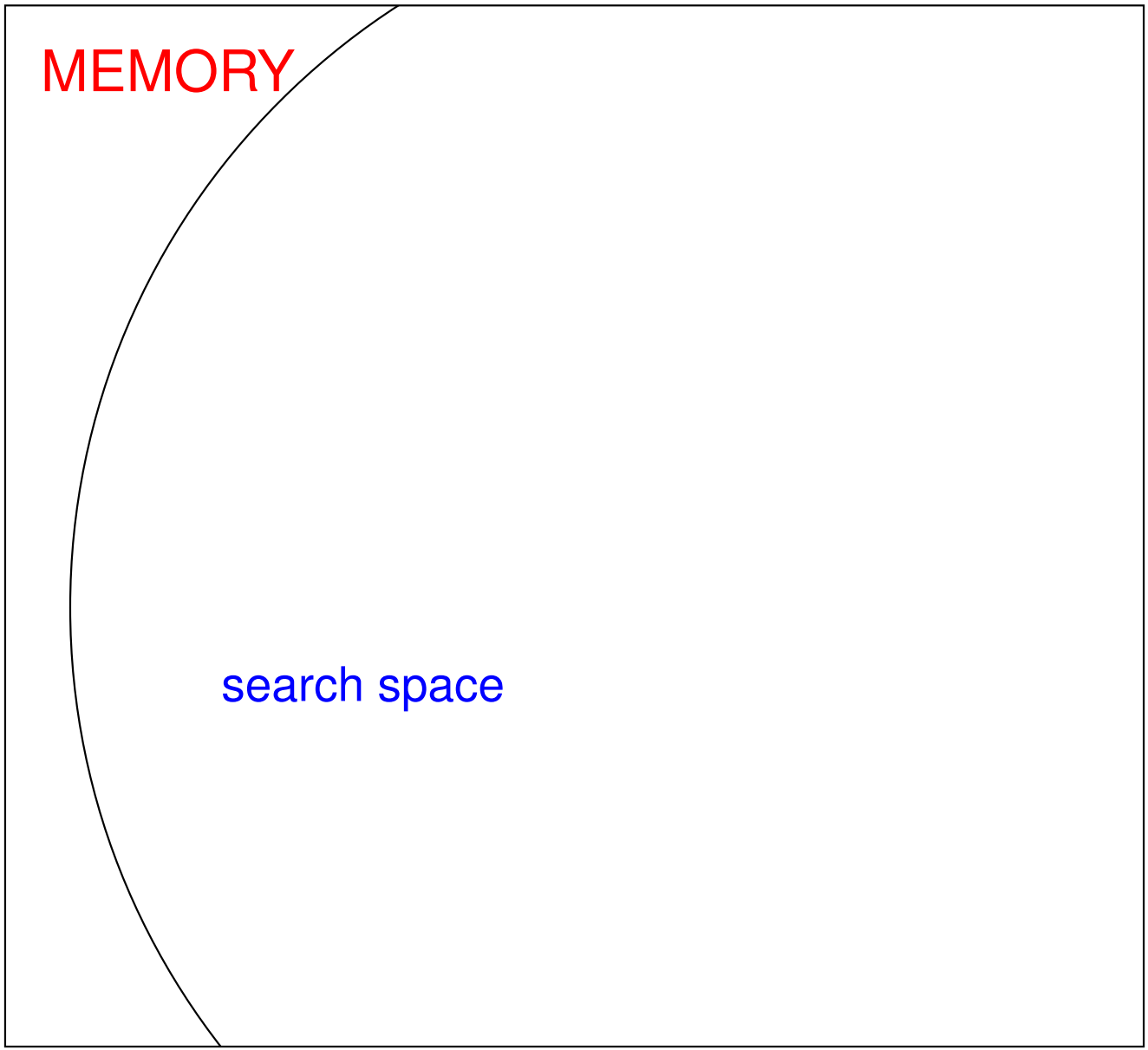

Saturation Algorithm

A saturation algorithm tries to saturate a set of clauses with respect to a given inference system.
In theory there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run forever, but without generating $\square$. In this case the input set of clauses is satisfiable.

Saturation Algorithm in Practice

In practice there are three possible scenarios:

1. At some moment the empty clause $\square$ is generated, in this case the input set of clauses is unsatisfiable.
2. Saturation will terminate without ever generating $\square$, in this case the input set of clauses in satisfiable.
3. Saturation will run until we run out of resources, but without generating $\square$. In this case it is unknown whether the input set is unsatisfiable.

Saturation Algorithm

Even when we implement inference selection by clause selection, there are too many inferences, especially when the search space grows.

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Solution: only apply inferences to the selected clause and the previously selected clauses.

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Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

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Thus, the search space is divided in two parts:

- active clauses, that participate in inferences;
- passive clauses, that do not participate in inferences.

Observation: the set of passive clauses is usually considerably larger than the set of active clauses, often by 2-4 orders of magnitude (depending on the saturation algorithm and the problem).

Outline

Subsumption and Tautology Deletion

A clause is a propositional tautology if it is of the form $A \vee \neg A \vee C$, that is, it contains a pair of complementary literals.
There are also equational tautologies, for example $a \not\simeq b \vee b \not\simeq c \vee f(c, c) \simeq f(a, a)$.

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A clause C subsumes any clause $C \vee D$, where D is non-empty.

It was known since 1965 that subsumed clauses and propositional tautologies can be removed from the search space.

Problem

How can we prove that completeness is preserved if we remove subsumed clauses and tautologies from the search space?

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Solution: general theory of redundancy.

Bag Extension of an Ordering

Bag = finite multiset.

Let $>$ be any ordering on a set X. The bag extension of $>$ is a binary relation $>^{bag}$, on bags over X, defined as the smallest transitive relation on bags such that

$$\{x, y_1, \ldots, y_n\} >^{bag} \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$$
$$\text{if } x > x_i \text{ for all } i \in \{1 \ldots m\},$$

where $m \geq 0$.

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where $m \geq 0$.
Idea: a bag becomes smaller if we replace an element by any finite number of smaller elements.
The following results are known about the bag extensions of orderings:

1. $>^{bag}$ is an ordering;
2. If $>$ is total, then so is $>^{bag}$;
3. If $>$ is well-founded, then so is $>^{bag}$.

Clause Orderings

From now on consider clauses also as bags of literals. Note:

- we have an ordering $\succ$ for comparing literals;
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Hence

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For simpicity we denote the multiset ordering also by $\succ$.

Redundancy

A clause $C \in S$ is called redundant in S if it is a logical consequence of clauses in S strictly smaller than C.

Examples

A tautology $A \vee \neg A \vee C$ is a logical consequence of the empty set of formulas:

$$\models A \vee \neg A \vee C,$$

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$$C \models C \vee D$$

therefore subsumed clauses are redundant.

If $\square \in S$, then all non-empty other clauses in S are redundant.

Redundant Clauses Can be Removed

In $\mathbb{BR}_\sigma$ (and in all calculi we will consider later) redundant clauses can be removed from the search space.

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Inference Process with Redundancy

Let $\mathbb{I}$ be an inference system. Consider an inference process with two kinds of step $S_i \Rightarrow S_{i+1}$:

1. Adding the conclusion of an $\mathbb{I}$-inference with premises in S_i.
2. Deletion of a clause redundant in S_i, that is

$$S_{i+1} = S_i - \{C\},$$

where C is redundant in S_i.

Fairness: Persistent Clauses and Limit

Consider an inference process

$$S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$$

A clause C is called persistent if

$$\exists i \forall j \geq i (C \in S_j).$$

The limit S_ω of the inference process is the set of all persistent clauses:

$$S_\omega = \bigcup_{i=0,1,\ldots} \bigcap_{j \geq i} S_j.$$

Fairness

The process is called $\mathbb{I}$-fair if every inference with persistent premises in S_ω has been applied, that is, if

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

is an inference in $\mathbb{I}$ and $\{C_1, \ldots, C_n\} \subseteq S_\omega$, then $C \in S_i$ for some i.

Completeness of $\mathbb{BR}_{\succ,\sigma}$

Completeness Theorem. Let $\succ$ be a simplification ordering and σ a well-behaved selection function. Let also

1. S_0 be a set of clauses;
2. $S_0 \Rightarrow S_1 \Rightarrow S_2 \Rightarrow \ldots$ be a fair $\mathbb{BR}_{\succ,\sigma}$-inference process.

Then S_0 is unsatisfiable if and only if $\square \in S_i$ for some i.

Saturation up to Redundancy

A set S of clauses is called saturated up to redundancy if for every $\mathbb{I}$-inference

$$\frac{C_1 \quad \ldots \quad C_n}{C}$$

with premises in S, either

1. $C \in S$; or
2. C is redundant w.r.t. S, that is, $S_{\prec C} \models C$.

End of Lecture 2

Slides for lecture 2 ended here . . .

Proof of Completeness

A trace of a clause C: a set of clauses $\{C_1, \ldots, C_n\} \subseteq S_\omega$ such that

1. $C \succ C_i$ for all $i = 1, \ldots, n$;
2. $C_1, \ldots, C_n \models C$.

Lemma 1. Every removed clause has a trace.
Lemma 2. The limit S_ω is saturated up to redundancy.
Lemma 3. The limit S_ω is logically equivalent to the initial set S_0.
Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\square \in S$.

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Interestingly, only the last lemma uses rules of $\mathbb{BR}_{\succ,\sigma}$.

Binary Resolution with Selection

One of the key properties to satisfy this lemma is the following: the conclusion of every rule is strictly smaller that the rightmost premise of this rule.

- Binary resolution,

$$\frac{\underline{p} \vee C_1 \quad \underline{\neg p} \vee C_2}{C_1 \vee C_2} \text{ (BR).}$$

- Positive factoring,

$$\frac{\underline{p} \vee \underline{p} \vee C}{p \vee C} \text{ (Fact).}$$

Saturation up to Redundancy and Satisfiability Checking

Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\square \in S$.

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Lemma 4. A set S of clauses saturated up to redundancy in $\mathbb{BR}_{\succ,\sigma}$ is unsatisfiable if and only if $\square \in S$.

Therefore, if we built a set saturated up to redundancy, then the initial set S_0 is satisfiable. This is a powerful way of checking redundancy: one can even check satisfiability of formulas having only infinite models.

The only problem with this characterisation is that there is no obvious way to build a model of S_0 out of a saturated set.

Outline

First-order logic with equality

- Equality predicate: $=$.
- Equality: $l = r$.

The order of literals in equalities does not matter, that is, we consider an equality $l = r$ as a multiset consisting of two terms l, r, and so consider $l = r$ and $r = l$ equal.

Equality. An Axiomatisation

- reflexivity axiom: $x = x$;
- symmetry axiom: $x = y \to y = x$;
- transitivity axiom: $x = y \wedge y = z \to x = z$;
- function substitution axioms:
  $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \to f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$, for every function symbol f;
- predicate substitution axioms:
  $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge P(x_1, \ldots, x_n) \to P(y_1, \ldots, y_n)$ for every predicate symbol P.

Inference systems for logic with equality

We will define a resolution and superposition inference system. This system is complete. One can eliminate redundancy (but the literal ordering needs to satisfy additional properties).

Inference systems for logic with equality

We will define a resolution and superposition inference system. This system is complete. One can eliminate redundancy (but the literal ordering needs to satisfy additional properties).
Moreover, we will first define it only for ground clauses. On the theoretical side,

- Completeness is first proved for ground clauses only.
- It is then “lifted” to arbitrary clauses using a technique called lifting.
- Moreover, this way some notions (ordering, selection function) can first be defined for ground clauses only and then it is relatively easy to see how to generalise them for non-ground clauses.

Simple Ground Superposition Inference System

Superposition: (right and left)

$$\frac{l = r \vee C \quad s[l] = t \vee D}{s[r] = t \vee C \vee D}\ (\text{Sup}), \quad \frac{l = r \vee C \quad s[l] \neq t \vee D}{s[r] \neq t \vee C \vee D}\ (\text{Sup}),$$

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$$\frac{s = t \vee s = t' \vee C}{s = t \vee t \neq t' \vee C}\ (\text{EF}),$$

Example

$$f(a) = a \vee g(a) = a$$
$$f(f(a)) = a \vee g(g(a)) \not\approx a$$
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Can this system be used for efficient theorem proving?

Not really. It has too many inferences. For example, from the clause $f(a) = a$ we can derive any clause of the form

$$f^m(a) = f^n(a)$$

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Worst of all, the derived clauses can be much larger than the original clause $f(a) = a$.
The recipe is to use the previously introduced ingredients:

1. Ordering;
2. Literal selection;
3. Redundancy elimination.

Atom and literal orderings on equalities

Equality atom comparison treats an equality $s = t$ as the multiset $\dot{\{}s, t\dot{\}}$.

- $(s' = t') \succ_{lit} (s = t)$ if $\dot{\{}s', t'\dot{\}} \succ \dot{\{}s, t\dot{\}}$.
- $(s' \not\approx t') \succ_{lit} (s \not\approx t)$ if $\dot{\{}s', t'\dot{\}} \succ \dot{\{}s, t\dot{\}}$.

Finally, we assert that all non-equality literals be greater than all equality literals.

Ground Superposition Inference System $\mathbb{S}\mathrm{up}_{\succ,\sigma}$

Let σ be a literal selection function.
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where (i) $l \succ r$, (ii) $s[l] \succ t$, (iii) $l = r$ is strictly greater than any literal in C, (iv) $s[l] = t$ is greater than or equal to any literal in D.

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where (i) $s \succ t \succeq t'$; (ii) $s = t$ is greater than or equal to any literal in C.

Extension to arbitrary (non-equality) literals

- Consider a two-sorted logic in which equality is the only predicate symbol.
- Interpret terms as terms of the first sort and non-equality atoms as terms of the second sort.
- Add a constant $\top$ of the second sort.
- Replace non-equality atoms $p(t_1, \ldots, t_n)$ by equalities of the second sort $p(t_1, \ldots, t_n) = \top$.

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For example, the clause

$$p(a, b) \vee \neg q(a) \vee a \neq b$$

becomes

$$p(a, b) = \top \vee q(a) \neq \top \vee a \neq b.$$

Binary resolution inferences can be represented by inferences in the superposition system

We ignore selection functions.

$$\frac{A \vee C_1 \quad \neg A \vee C_2}{C_1 \vee C_2}\ (\mathrm{BR})$$

$$\frac{\dfrac{A = \top \vee C_1 \quad A \not\approx \top \vee C_2}{\top \not\approx \top \vee C_1 \vee C_2}\ (\mathrm{Sup})}{C_1 \vee C_2}\ (\mathrm{ER})$$

Exercise

Positive factoring can also be represented by inferences in the superposition system.

Simplification Ordering

The only restriction we imposed on term orderings was well-foundedness and stability under substitutions. When we deal with equality, these two properties are insufficient. We need a third property, called monotonicity.
An ordering $\succ$ on terms is called a simplification ordering if

1. $\succ$ is well-founded;
2. $\succ$ is monotonic: if $l \succ r$, then $s[l] \succ s[r]$;
3. $\succ$ is stable under substitutions: if $l \succ r$, then $l\theta \succ r\theta$.

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One can combine the last two properties into one:

2a. If $l \succ r$, then $s[l\theta] \succ s[r\theta]$.

End of Lecture 3

Slides for lecture 3 ended here . . .

A General Property of Term Orderings

If $\succ$ is a simplification ordering, then for every term $t[s]$ and its proper subterm s we have $s \not\succ t[s]$.

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Consider an example.

$$f(a) = a$$
$$f(f(a)) = a$$
$$f(f(f(a))) = a$$

Then both $f(f(a)) = a$ and $f(f(f(a))) = a$ are redundant. The clause $f(a) = a$ is a logical consequence of $\{f(f(a)) = a, f(f(f(a))) = a\}$ but is not redundant.

Term Algebra

Term algebra $TA(\Sigma)$ of signature Σ:

- Domain: the set of all ground terms of Σ.
- Interpretation of any function symbol f or constant c is defined as follows::

$$f_{TA(\Sigma)}(t_1, \ldots, t_n) \stackrel{\text{def}}{\Leftrightarrow} f(t_1, \ldots, t_n);$$
$$c_{TA(\Sigma)} \stackrel{\text{def}}{\Leftrightarrow} c.$$

Knuth-Bendix Ordering, Ground Case

Let us fix

- Signature Σ, it induces the term algebra $TA(\Sigma)$.
- Total ordering $\gg$ on Σ, called precedence relation;
- Weight function $w : \Sigma \to \mathbb{N}$.

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$$|g(t_1, \ldots, t_n)| = w(g) + \sum_{i=1}^{n} |t_i|.$$

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Example

$$\begin{aligned} w(a) &= 1 \\ w(b) &= 2 \\ w(f) &= 3 \\ w(g) &= 0 \end{aligned}$$

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There exists also a non-ground version of the Knuth-Bendix ordering and a (nearly) linear time algorithm for term comparison using this ordering.
The Knuth-Bendix ordering is the main ordering used on Vampire and all other resolution and superposition theorem provers.

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Same Property

The conclusion is strictly smaller than the rightmost premise:

$$\frac{\underline{l = r} \vee C \quad s[\underline{l}] = t \vee D}{s[r] = t \vee C \vee D}\ (\text{Sup}), \qquad \frac{\underline{l = r} \vee C \quad s[\underline{l}] \not\approx t \vee D}{s[r] \not\approx t \vee C \vee D}\ (\text{Sup}),$$

where (i) $l \succ r$, (ii) $s[l] \succ t$, (iii) $l = r$ is strictly greater than any literal in C, (iv) $s[l] = t$ is greater than or equal to any literal in D.

New redundancy

Consider a superposition with a unit left premise:

$$\frac{\underline{l = r} \quad \underline{s[l] = t \vee D}}{s[r] = t \vee D}\ \text{(Sup)},$$

Note that we have

$$l = r, s[r] = t \vee D \models s[l] = t \vee D$$

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This rule (superposition plus deletion) is sometimes called demodulation (also rewriting by unit equalities).

Outline

Substitution

- A substitution θ is a mapping from variables to terms such that the set $\{x \mid \theta(x) \neq x\}$ is finite.
- This set is called the domain of θ.
- Notation: $\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where $x_1, \ldots, x_n$ are pairwise different variables, denotes the substitution θ such that

$$\theta(x) = \begin{cases} t_i & \text{if } x = x_i; \\ x & \text{if } x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

- Application of this substitution to an expression E: simultaneous replacement
of x_i by t_i.
- Application of a substitution θ to E is denoted by $E\theta$.
- Since substitutions are functions, we can define their composition (written $\sigma\tau$ instead of $\tau \circ \sigma$). Note that we have $E(\sigma\tau) = (E\sigma)\tau$.

Exercise

Exercise: Suppose we have two substitutions

$$\{x_1 \mapsto s_1, \ldots, x_m \mapsto s_m\} \text{ and}$$
$$\{y_1 \mapsto t_1, \ldots, y_n \mapsto t_n\}.$$

How can we write their composition using the same notation?

Instances, Ground

An instance of an expression (that is term, atom, literal, or clause) E is obtained by applying a substitution to E. Examples:

- some instances of the term $f(x, a, g(x))$ are:
  $f(x, a, g(x))$,
  $f(y, a, g(y))$,
  $f(a, a, g(a))$,
  $f(g(b), a, g(g(b)))$;
- but the term $f(b, a, g(c))$ is not an instance of this term.

Ground instance: instance with no variables.

Herbrand's Theorem

For a set of clauses S denote by S^* the set of ground instances of clauses in S.

Theorem Let S be a set of clauses. The following conditions are equivalent.

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The theorem reduces the problem of checking unsatisfiability of sets of arbitrary clauses to checking unsatisfiability of sets of ground clauses . . .
The only problem is that S^* can be infinite even if S is finite.

Note on Herbrand's Theorem, Compactness and Completeness

The proofs of completeness of resolution and superposition with redundancy elimination does not use any of these theorems.

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The proofs of completeness of resolution and superposition with redundancy elimination does not use any of these theorems.

Interestingly, they all can be derived as simple corollaries of this proof of completeness!

Lifting

Lifting is a technique for proving completeness theorems in the following way:

1. Prove completeness of the system for a set of ground clauses;
2. Lift the proof to the non-ground case.

Lifting, Example

Consider two (non-ground) clauses $p(x, a) \vee q_1(x)$ and $\neg p(y, z) \vee q_2(y, z)$. If the signature contains function symbols, then both clauses have infinite sets of instances:

$$\begin{array}{rcl} \{p(r, a) \vee q_1(r) & \mid & r \text{ is ground}\} \\ \{\neg p(s, t) \vee q_2(s, t) & \mid & s, t \text{ are ground}\} \end{array}$$

We can resolve such instances if and only if $r = s$ and $t = a$. Then we can apply the following inference

$$\frac{p(s, a) \vee q_1(s) \quad \neg p(s, a) \vee q_2(s, a)}{q_1(s) \vee q_2(s, a)} \text{ (BR)}$$

But there is an infinite number of such inferences.

Lifting, Idea

The idea is to represent an infinite number of ground inferences of the form

$$\frac{p(s,a) \vee q_1(s) \quad \neg p(s,a) \vee q_2(s,a)}{q_1(s) \vee q_2(s,a)} \text{ (BR)}$$

by a single non-ground inference

$$\frac{p(x,a) \vee q_1(x) \quad \neg p(y,z) \vee q_2(y,z)}{q_1(y) \vee q_2(y,a)} \text{ (BR)}$$

Is this always possible?

Yes!

$$\frac{p(x,a) \vee q_1(x) \quad \neg p(y,z) \vee q_2(y,z)}{q_1(y) \vee q_2(y,a)} \text{ (BR)}$$

Note that the substitution $\{x \mapsto y, z \mapsto a\}$ is a solution of the "equation" $p(x,a) = p(y,z)$.

What should we lift?

- Ordering $\succ$;
- Selection function σ;
- Calculus $\mathbb{S}\mathrm{up}_{\succ,\sigma}$.

Most importantly, for the lifting to work we should be able to solve equations $s \simeq t$ between terms and between atoms. This can be done using most general unifiers.

Unifier

Unifier of expressions s_1 and s_2: a substitution θ such that $s_1\theta = s_2\theta$. In other words, a unifier is a solution to an "equation" $s_1 = s_2$. In a similar way we can define solutions to systems of equations $s_1 = s'_1, \ldots, s_n = s'_n$. We call such solutions simultaneous unifiers of $s_1, \ldots, s_n$ and $s'_1, \ldots, s'_n$.

(Most General) Unifiers

A solution θ to a set of equations E is said to be a most general solution if for every other solution σ there exists a substitution τ such that $\theta\tau = \sigma$. In a similar way can define a most general unifier.

(Most General) Unifiers

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Consider terms $f(x_1, g(x_1), x_2)$ and $f(y_1, y_2, y_2)$.
(Some of) their unifiers are
$\theta_1 = \{y_1 \mapsto x_1, y_2 \mapsto g(x_1), x_2 \mapsto g(x_1)\}$ and
$\theta_2 = \{y_1 \mapsto a, y_2 \mapsto g(a), x_2 \mapsto g(a), x_1 \mapsto a\}$:

$f(x_1, g(x_1), x_2)\theta_1 = f(x_1, g(x_1), g(x_1))$;
$f(y_1, y_2, y_2)\theta_1 = f(x_1, g(x_1), g(x_1))$;
$f(x_1, g(x_1), x_2)\theta_2 = f(a, g(a), g(a))$;
$f(y_1, y_2, y_2)\theta_2 = f(a, g(a), g(a))$.

But only θ_1 is most general.

Unification

Let E be a set of equations. An isolated equation in E is any equation $x = t$ in it such that x has exactly one occurrence in E.

input:
A finite set of equations E
output:
A solution to E or failure.
begin
while there exists a non-isolated equation $(s = t) \in E$
do
case (s, t) **of**
$(t, t) \Rightarrow$ Remove this equation from E
$(x, t) \Rightarrow$
if x occurs in t
then halt with failure
else replace x by t in all other equations of E
$(t, x) \Rightarrow$ replace this equation by $x = t$
and do the same as in the case (x, t)
$(c, d) \Rightarrow$ halt with failure
$(c, f(t_1, \ldots, t_n)) \Rightarrow$ halt with failure
$(f(t_1, \ldots, t_n), c) \Rightarrow$ halt with failure
$(f(s_1, \ldots, s_m), g(t_1, \ldots, t_n)) \Rightarrow$ halt with failure
$(f(s_1, \ldots, s_n), f(t_1, \ldots, t_n)) \Rightarrow$ replace this equation by the set
$s_1 = t_1, \ldots, s_n = t_n$
end
od
Now E has the form $\{x_1 = r_1, \ldots, x_l = r_l\}$ and every equation in it is isolated
return the substitution $\{x_1 \mapsto r_1, \ldots, x_l \mapsto r_l\}$
end

Examples

$$\{h(g(f(x), a)) = h(g(y, y))\}$$
$$\{h(f(y), y, f(z)) = h(z, f(x), x)\}$$
$$\{h(g(f(x), z)) = h(g(y, y))\}$$

Properties

Theorem Suppose we run the unification algorithm on $s = t$. Then

- If s and t are unifiable, then the algorithms terminates and outputs a most general unifier of s and t.
- If s and t are not unifiable, then the algorithms terminates with failure.

Notation (slightly ambiguous):

- $mgu(s, t)$ for a most general unifier;
- $mgs(E)$ for a most general solution.

Exercise

Consider a trivial system of equations $\{\}$ or $\{a = a\}$.
What is the set of solutions to it?
What is the set of most general solutions to it?

Properties

Theorem Let C be a clause and E a set of equations. Then

$$\{D \in C^* \mid \exists\theta(C\theta = D \text{ and } \theta \text{ is a solution to } E)\} = (Cmgs(E))^*.$$

In other words, to find a set of ground instances of a clause C that also satisfy an equation E, take the most general solution σ of E and use ground instances of $C\sigma$.

Non-Ground Superposition Rule

Superposition:

$$\frac{l = r \vee C \quad s[l'] = t \vee D}{(s[r] = t \vee C \vee D)\theta}\ (\text{Sup}), \qquad \frac{l = r \vee C \quad s[l'] \neq t \vee D}{(s[r] \neq t \vee C \vee D)\theta}\ (\text{Sup}),$$

where

1. θ is an mgu of l and l';
2. l' is not a variable;
3. $r\theta \not\succeq l\theta$;
4. $t\theta \not\succeq s[l']\theta$.
5. ...

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where

1. θ is an mgu of l and l';
2. l' is not a variable;
3. $r\theta \not\succeq l\theta$;
4. $t\theta \not\succeq s[l']\theta$.
5. ...

Observations:

- ordering is partial, hence conditions like $r\theta \not\succeq l\theta$;
- these conditions must be checked a posteriori, that is, after the rule has been applied.

Note, however, that $l \succ r$ implies $l\theta \succ r\theta$, so checking orderings a priory helps.

More rules

Equality Resolution:

$$\frac{s \neq s' \vee C}{C\theta} \text{ (ER)},$$

where θ is an mgu of s and s'.

Equality Factoring:

$$\frac{l = r \vee l' = r' \vee C}{(l = r \vee r \neq r' \vee C)\theta} \text{ (EF)},$$

where θ is an mgu of l and l', $r\theta \not\succeq l\theta$, $r'\theta \not\succeq l\theta$, and $r'\theta \not\succeq r\theta$.

Outline

From theory to practice

- Preprocessing and CNF transformation;
- Superposition system;
- Orderings;
- Selection functions;
- Fairness (saturation algorithms);
- Redundancy.

Vampire's preprocessing (incomplete list)

1. (Optional) Select a relevant subset of formulas.
2. (Optional) Add theory axioms;
3. Rectify the formula.
4. If the formula contains any occurrence of $\top$ or $\bot$, simplify the formula.
5. Remove if-then-else and let-in connectives.
6. Flatten the formula.
7. Apply pure predicate elimination.
8. (Optional) Remove unused predicate definitions.
9. Convert the formula into equivalence negation normal form.
10. Use a naming technique to replace some subformulas by their names.
11. Convert the formula into negation normal form.
12. Skolemize the formula.
13. (Optional) Replace equality axioms.
14. Determine a literal ordering to be used.
15. Transform the formula into its conjunctive normal form.
16. (Optional) Function definition elimination.
17. (Optional) Inequality splitting.
18. Remove tautologies.
19. Pure literal elimination.
20. Remove clausal definitions.

Checking Redundancy

Suppose that the current search space S contains no redundant clauses. How can a redundant clause appear in the inference process?

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Only when a new clause (a child of the selected clause and possibly other clauses) is added.
Classification of redundancy checks:

- The child is redundant;
- The child makes one of the clauses in the search space redundant.

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Classification of redundancy checks:

- The child is redundant;
- The child makes one of the clauses in the search space redundant.

We use some fair strategy and perform these checks after every inference that generates a new clause.
In fact, one can do better.

Demodulation, Non-Ground Case

$$\frac{l = r \quad L[l'] \vee D}{L[r\theta] \vee D} \text{ (Dem)},$$

where $l\theta = l'$, $l\theta \succ r\theta$, and $(L[l'] \vee D)\theta \succ (l\theta \succ r\theta)$.

Demodulation, Non-Ground Case

$$\frac{l = r \quad L[l'] \vee D}{L[r\theta] \vee D} \text{ (Dem)},$$

where $l\theta = l'$, $l\theta \succ r\theta$, and $(L[l'] \vee D)\theta \succ (l\theta \succ r\theta)$.

Easier to understand:

$$\frac{l = r \quad L[l\theta] \vee D}{L[r\theta] \vee D} \text{ (Dem)},$$

where $l\theta \succ r\theta$, and $(L[l'] \vee D)\theta \succ (l\theta \succ r\theta)$.

Generating and Simplifying Inferences

An inference

$$\frac{C_1 \quad \ldots \quad C_n}{C} .$$

is called simplifying if at least one premise C_i becomes redundant after the addition of the conclusion C to the search space. We then say that C_i is simplified into C.
A non-simplifying inference is called generating.

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Note. The property of being simplifying is undecidable. So is the property of being redundant. So in practice we employ sufficient conditions for simplifying inferences and for redundancy.

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A non-simplifying inference is called generating.

Note. The property of being simplifying is undecidable. So is the property of being redundant. So in practice we employ sufficient conditions for simplifying inferences and for redundancy.

Idea: try to search eagerly for simplifying inferences bypassing the strategy for inference selection.

Generating and Simplifying Inferences

Two main implementation principles:

apply simplifying inferences eagerly;
apply generating inferences lazily.

checking for simplifying inferences should pay off;
so it must be cheap.

End of Lecture 4

Slides for lecture 4 ended here . . .

Redundancy Checking

Redundancy-checking occurs upon addition of a new child C. It works as follows

- Retention test: check if C is redundant.
- Forward simplification: check if C can be simplified using a simplifying inference.
- Backward simplification: check if C simplifies or makes redundant an old clause.

Examples

Retention test:

- tautology-check;
- subsumption.

(A clause C subsumes a clause D if there exists a substitution θ such that $C\theta$ is a submultiset of D.)

Simplification:

- demodulation (forward and backward);
- subsumption resolution (forward and backward).

Some redundancy criteria are expensive

- Tautology-checking is based on congruence closure.
- Subsumption and subsumption resolution are NP-complete.

How can one efficiently apply complex operations to hundreds of thousands of terms and clauses?

Term Indexing

How can one efficiently apply complex operations to hundreds of thousands of terms and clauses?
Given a set $\mathcal{L}$ (the set of indexed terms), a binary relation R over terms (the retrieval condition) and a term t (called the query term), identify the subset $\mathcal{M}$ of $\mathcal{L}$ consisting of all of the terms l such that $R(l, t)$ holds.

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The problem (and solution) is similar to database query answering, but data are much more complex than relational data (a clause is a finite set of trees, so the search space is a (large) set of finite sets of trees).

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The problem (and solution) is similar to database query answering, but data are much more complex than relational data (a clause is a finite set of trees, so the search space is a (large) set of finite sets of trees).
One puts the clauses in $\mathcal{L}$ in a data structure, called the index. The data structure is designed with the only purpose to make the retrieval fast.

Term Indexing

- Different indexes are needed to support different operations;
- The set of clauses is dynamically (and often) changes, so that index maintenance must be efficient.
- Memory is an issue (badly designed indexes may take much more space than clauses).
- The inverse retrieval conditions (the same algorithm on clauses) may require very different indexing techniques (e.g., forward and backward subsumption).
- Sensitive to the signature of the problem: techniques good for small signatures are too slow and too memory consuming for large signatures.

Term Indexing in Vampire

- Various hash tables.
- Flatterms in constant memory for storing temporary clauses.
- Code trees for forward subsumption;
- Code trees with precompiled ordering constraints;
- Discrimination trees;
- Substitution trees;
- Variables banks;
- Shared terms with renaming lists;
- Path index with compiled database joins;
- ...

Observations

- There may be chains (repeated applications) of forward simplifications.
- After a chain of forward simplifications another retention test can (should) be done.

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- After a chain of forward simplifications another retention test can (should) be done.
- Backward simplification is often expensive.
- In practice, the retention test may include other checks, resulting in the loss of completeness, for example, we may decide to discard too heavy clauses.

How to Design a Good Saturation Algorithm?

A saturation algorithm must be fair: every possible generating inference must eventually be selected.

Two main implementation principles:

apply simplifying inferences eagerly;
apply generating inferences lazily.

checking for simplifying inferences should pay off;
so it must be cheap.

Given Clause Algorithm (no Simplification)

input: *init*: set of clauses**;**
var *active*, *passive*, *queue*: sets of clauses**;**
var *current*: clauses **;**
active $:= \emptyset$**;**
passive $:=$ *init***;**
while *passive* $\neq \emptyset$ **do**
* *current* $:=$ *select*(*passive*)**;** (* clause selection *)
move *current* from *passive* to *active*;
* *queue* $:=$*infer*(*current*, *active*)**;** (* generating inferences *)
if $\square \in$ *queue* **then return** *unsatisfiable***;**
passive $:=$ *passive* $\cup$ *queue*
od;
return *satisfiable*

Given Clause Algorithm (with Simplification)

In fact, there is more than one . . .

Otter vs. Discount Saturation

Otter saturation algorithm:

- active clauses participate in generating and simplifying inferences;
- passive clauses participate in simplifying inferences.

Discount saturation algorithm:

- active clauses participate in generating and simplifying inferences;
- passive clauses do not participate in inferences.

Otter vs. Discount Saturation, Newly Generated Clauses

Otter saturation algorithm:

- active clauses participate in generating and simplifying inferences;
- new clauses participate in simplifying inferences;
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- new clauses participate in simplifying inferences;
- passive clauses do not participate in inferences.

Otter vs. Discount Saturation, Newly Generated Clauses

Otter saturation algorithm:

- active clauses participate in generating inferences with the selected clause and simplifying inferences with new clauses;
- new clauses participate in simplifying inferences with all clauses;
- passive clauses participate in simplifying inferences with new clauses.

Discount saturation algorithm:

- active clauses participate in generating inferences and simplifying inferences with the selected clause and simplifying inferences with the new clauses;
- new clauses participate in simplifying inferences with the selected and active clauses;
- passive clauses do not participate in inferences.

Otter Saturation Algorithm

```
    input: init: set of clauses;
    var active, passive, unprocessed: set of clauses;
    var given, new: clause;
    active := ∅;
    unprocessed := init;
    loop
      while unprocessed ≠ ∅
        new :=pop(unprocessed);
        if new = □ then return unsatisfiable;
*       if retained(new) then                          (* retention test *)
*         simplify new by clauses in active ∪ passive ;(* forward simplification *)
          if new = □ then return unsatisfiable;
*         if retained(new) then                        (* another retention test *)
*           delete and simplify clauses in active and (* backward simplification *)
                                              passive using new;
            move the simplified clauses to unprocessed;
            add new to passive
      if passive = ∅ then return satisfiable or unknown
*     given := select(passive);                        (* clause selection *)
      move given from passive to active;
*     unprocessed :=infer(given, active);              (* generating inferences *)
```

Discount Saturation Algorithm

```
   input: init: set of clauses;
   var active, passive, unprocessed: set of clauses;
   var given, new: clause;
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*              delete and simplify clauses in active using new;(* backward simplification *)
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*     given := select(passive);                                  (* clause selection *)
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      if retained(given) then                                   (* retention test *)
*        delete and simplify clauses in active using given;     (* backward simplification *)
         move the simplified clauses to unprocessed;
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Age-Weight Ratio

How to select nice clauses?

- Small clauses are nice.
- Selecting only small clauses can postpone the selection of an old clause (e.g., input clause) for too long, in practice resulting in incompleteness.

Age-Weight Ratio

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- Small clauses are nice.
- Selecting only small clauses can postpone the selection of an old clause (e.g., input clause) for too long, in practice resulting in incompleteness.

Solution:

- A fixed percentage of clauses is selected by weight, the rest are selected by age.
- So we use an age-weight ratio $a : w$: of each $a + w$ clauses select a oldest and w smallest clauses.

Limited Resource Strategy

Limited Resource Strategy: try to approximate which clauses are unreachable by the end of the time limit and remove them from the search space.

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Try:

```
vampire --age_weight_ratio 10:1
  --backward_subsumption off
  --time_limit 86400
      GRP140-1.p
```

Outline

Interpolation

Theorem

Let A, B be closed formulas and let $A \vdash B$.

Then there exists a formula I such that

1. $A \vdash I$ *and* $I \vdash B$*;*
2. *every symbol of I occurs both in A and B;*

Interpolation

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Then there exists a formula I such that

1. *$A \vdash I$ and $I \vdash B$;*
2. *every symbol of I occurs both in A and B;*

Any formula I with this property is called an interpolant of A and B.

Essentially, an interpolant is a formula that is

1. intermediate in power between A and B;
2. Uses only common symbols of A and B.

Interpolation has many uses in verification.

Interpolation

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Interpolation has many uses in verification.

When we deal with refutations rather than proofs and have an unsatisfiable set $\{A, B\}$, it is convenient to use reverse interpolants of A and B, that is, a formula I such that

1. $A \vdash I$ and $\{I, B\}$ is unsatisfiable;
2. every symbol of I occurs both in A and B;

Interpolation Through Colors

- There are three colors: blue, red and green.

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- We have two formulas: A and B.
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Interpolation Through Colors

- There are three colors: blue, red and green.
- Each symbol (function or predicate) is colored in exactly one of these colors.
- We have two formulas: A and B.
- Each symbol in A is either blue or green.
- Each symbol in B is either red or green.
- We know that $\vdash A \to B$.
- Our goal is to find a green formula I such that
  1. $\vdash A \to I$;
  2. $\vdash I \to B$.

Interpolation with Theories

- Theory T: any set of closed green formulas.
- $C_1, \ldots, C_n \vdash_T C$ denotes that the formula $C_1 \wedge \ldots \wedge C_1 \rightarrow C$ holds in all models of T.
- Interpreted symbols: symbols occurring in T.
- Uninterpreted symbols: all other symbols.

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- Interpreted symbols: symbols occurring in T.
- Uninterpreted symbols: all other symbols.

Theorem

Let A, B be formulas and let $A \vdash_T B$.

Then there exists a formula I such that

1. *$A \vdash_T I$ and $I \vdash B$;*
2. *every uninterpreted symbol of I occurs both in A and B;*
3. *every interpreted symbol of I occurs in B.*

Likewise, there exists a formula I such that

1. *$A \vdash I$ and $I \vdash_T B$;*
2. *every uninterpreted symbol of I occurs both in A and B;*
3. *every interpreted symbol of I occurs in A.*

Local Derivations

A derivation is called local (well-colored) if each inference in it

$$\frac{C_1 \quad \cdots \quad C_n}{C}$$

either has no blue symbols or has no red symbols.
That is, one cannot mix blue and red in the same inference.

Local Derivations: Example

- $A := \forall x(x = a)$
- $B := c = b$
- Interpolant: $\forall x \forall y(x = y)$ (note: universally quantified!)

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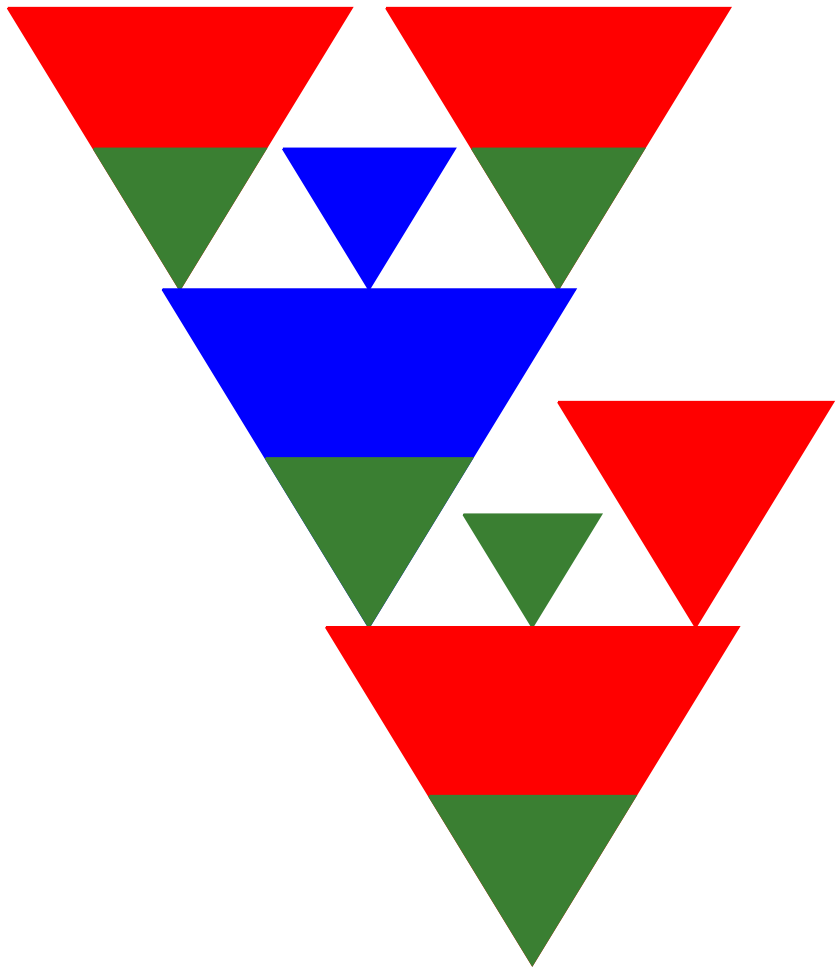
- $B := c = b$
- Interpolant: $\forall x \forall y(x = y)$ (note: universally quantified!)

A local refutation in the superposition calculus:

$$\dfrac{\dfrac{x = a \quad y = a}{x = y} \qquad c \neq b}{\dfrac{y \neq b}{\bot}}$$

Shape of a local derivation



Symbol Eliminating Inference

- At least one of the premises is not green.
- The conclusion is green.

$$\dfrac{\dfrac{x = a \quad y = a}{x = y} \qquad c \neq b}{\dfrac{y \neq b}{\bot}}$$

Extracting Interpolants from Local Proofs

Theorem

Let Π be a local refutation. Then one can extract from Π in linear time a reverse interpolant I of A and B. This interpolant is ground if all formulas in Π are ground.

Extracting Interpolants from Local Proofs

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Let Π be a local refutation. Then one can extract from Π in linear time a reverse interpolant I of A and B. This interpolant is ground if all formulas in Π are ground. This reverse interpolant is a boolean combination of conclusions of symbol-eliminating inferences of Π.

Extracting Interpolants from Local Proofs

Theorem

Let Π be a local refutation. Then one can extract from Π in linear time a reverse interpolant I of A and B. This interpolant is ground if all formulas in Π are ground. This reverse interpolant is a boolean combination of conclusions of symbol-eliminating inferences of Π.

What is remarkable in this theorem:

- No restriction on the calculus (only soundness required) – can be used with theories.
- Can generate interpolants in theories where no good interpolation algorithms exist.

Interpolation: Examples in Vampire

```
fof(fA,axiom, q(f(a)) & ~q(f(b)) ).
fof(fB,conjecture, ?[V]: V != c).
```

Interpolation: Examples in Vampire

```
% request to generate an interpolant
vampire(option,show_interpolant,on).
% symbol coloring
vampire(symbol,predicate,q,1,left).
vampire(symbol,function,f,1,left).
vampire(symbol,function,a,0,left).
vampire(symbol,function,b,0,left).
vampire(symbol,function,c,0,right).
% formula L
vampire(left_formula).
  fof(fA,axiom, q(f(a)) & ~q(f(b)) ).
vampire(end_formula).
% formula R
vampire(right_formula).
  fof(fB,conjecture, ?[V]: V != c).
vampire(end_formula).
```

Symbol Elimination

Colored proofs can also be used for an interesting application. Suppose that we have a set of formulas in some language and want to derive consequences of these formulas in a subset of this language.

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Then we declare the symbols to be eliminated colored and ask Vampire to output symbol-eliminating inferences.

This technique was used in our experiments on automatic loop invariant generation.

Outline

Sorts

Consider these statements:

1. Sort b consists of two elements: t and f;
2. Sort s has three different elements.

$$t! = f \wedge (\forall x : b)(x = t \vee x = f)$$
$$(\exists x : s)(\exists y : s)(\exists z : s)(x \neq y \wedge x \neq z \wedge y \neq z)$$

Sorts

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$$t! = f \wedge (\forall x : b)(x = t \vee x = f)$$
$$(\exists x : s)(\exists y : s)(\exists z : s)(x \not\approx y \wedge x \not\approx z \wedge y \not\approx z)$$

The unsorted version of it:

$$(\forall x)(x = t \vee x = f)$$
$$(\exists x)(\exists y)(\exists z)(x \not\approx y \wedge x \not\approx z \wedge y \not\approx z)$$

is unsatisfiable:

```
fof(1,axiom,t != f & ! [X] : X = t | X = f).
fof(1,axiom,? [X,Y,Z] : (X != Y & X != Z & Y != Z)).
```

```
vampire sort1.tptp
```

Sorts in TPTP

```
tff(boolean_type,type,b: $tType).    % b is a sort
tff(s_is_a_type,type,s: $tType).     % s is a sort

tff(t_has_type_b,type,t : b).    % t has sort b
tff(f_has_type_b,type,f : b).    % f has sort b

tff(1,axiom,t != f & ! [X:b] : X = t | X = f).
tff(1,axiom,? [X:s,Y:s,Z:s] : (X != Y & X != Z & Y != Z)).
```

```
vampire --splitting off
  --saturation_algorithm inst_gen sort2.tptp
```

Pre-existing sorts

- `$i`: sort of individuals. If is the default sort: if a symbol is not declared, it has this sort.
- `$int`: sort of integers.
- `$rat`: sort of rationals.
- `$real`: sort of reals.

Integers

One can use concrete integers and some interpreted functions on them.

```
tff(1,conjecture,$sum(2,2)=4).
```

```
vampire --inequality_splitting 0 int1.tptp
```

Interpreted Functions and Predicates on Integers

Functions:

- `$sum`: addition $(x + y)$
- `$product`: multiplication $(x \cdot y)$
- `$difference`: difference $(x - y)$
- `$uminus`: unary minus $(-x)$
- `$to_rat`: conversion to rationals.
- `$to_real`: conversion to reals.

Predicates:

- `$less`: less than $(x < y)$
- `$lesseq`: less than or equal to $(x \leq y)$
- `$greater`: greater than $(x > y)$
- `$greatereq`: greater than or equal to $(x \geq y)$

How Vampire Proves Problems in Arithmetic

- adding theory axioms;
- evaluating expressions, when possible;
- (future) SMT solving.

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Example:

$$(x + y) + z = x + (z + y).$$

```
tff(1,conjecture,
  ! [X:$int,Y:$int,Z:$int] :
    $sum($sum(X,Y),Z)=$sum(X,$sum(Z,Y))).
```

```
vampire --inequality_splitting 0 int2.tptp
```

How Vampire Proves Problems in Arithmetic

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```

```
vampire --inequality_splitting 0 int2.tptp
```

- You can add your own axioms;
- you can replace Vampire axioms by your own: use

  `--theory_axioms off`

Outline

CASC Mode

```
vampire --mode casc SET014-3.p
```

If-then-else and Let-in

A partial correctness statement:

```
{∀X(p(X) => X ≥ 0)}
{∀X(q(X) > 0)}
{p(a)}
if (r(a)) {
  a := a+1
}
else {
  a := a + q(a).
}
{a > 0}
```

If-then-else and Let-in

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The next state function for `a`:

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a' =
  if r(a)
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```

In Vampire:

```
tff(1,type,p : $int > $o).
tff(2,type,q : $int > $int).
tff(3,type,r : $int > $o).
tff(4,type,a : $int).

tff(5,hypothesis,! [X:$int] :
  (p(X) => $greatereq(X,0))).
tff(6,hypothesis,! [X:$int] :
  ($greatereq(q(X),0))).
tff(7,hypothesis,p(a)).

tff(8,hypothesis,
  a0 = $ite_t(r(a),
    $let_tt(a,$sum(a,1),a),
    $let_tt(a,$sum(a,q(a)),a)
)).

tff(9,conjecture,$greater(a0,0)).
```

Consequence Elimination

Given a large set of formulas, find out which formulas are consequences of other formulas in the set.
For example, used for pruning a set of automatically found loop invariants.

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```
fof(ax1, axiom,a => b).
fof(ax2, axiom,b => c).
fof(ax3, axiom,c => a).

fof(c1, claim, a | d).
fof(c2, claim, b | d).
fof(c3, claim, c | d).
```

```
vampire --mode consequence_elimination consequence.tptp
```

End of Lecture 5

Slides for lecture 5 ended here . . .